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Three Essays on Structural Models

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Abstract

My thesis includes three papers on contingent claims valuation of corporate securities using structural models of credit risk. Our study focuses on structural models and their applications in estimating damages in security class actions, option pricing and warrant pricing.

Securities class actions typically involve some misrepresentation by a firm that overstates its true value. In securities class actions econometric models are used to assess damages to shareholders. However, studies on measuring damages for debt-holders are limited. My first paper uses a modified Merton framework to measure the impact of misrepresentation on the value of other components (e.g., debt, warrants) of a firm's capital structure. Using structural models and leveraging the relationship between equity and firm value, we use observable equity information to determine firm value and hence the effect of misrepresentation on value of other securities in the capital structure. We investigate various capital structures and show that misrepresentation can have a significant impact on the value of all components in the capital structure. We find that the misrepresentation impact on debt value depends on firm leverage and debt seniority and not on the warrant dilution factor. Generally, the debt for higher-leverage firms is more sensitive to the misrepresentation impact than for lower-leverage firms and junior debt is more affected by fraud than senior debt. The impact on warrant value is determined by warrant moneyness (stock price), with the dilution factor having no effect.

My second paper extends the study in my first paper into the First Passage Time (FPT) framework, which is capable of modeling firms with complex debt structures. Our findings have important consequences for damages assessment and allocation of settlement awards in securities class actions. In some jurisdictions damages awarded are net of any hedge or risk-limitation transaction. Since corporate securities such as bonds and stocks are often held in portfolios for hedging purposes, measuring the effect of misrepresentation on all of the firm's issuances is essential to accurately computed damages awards. In addition to our main findings, we explicitly discuss bankruptcy costs for the First Passage Time model. Furthermore, we are able to reduce a system of two non-linear equations, used to connect the unobservable firm value and firm value volatility to observable equity value and equity volatility, into one equation. This technique improves the ability to solve the non-linear system.

My third paper studies option and warrant pricing under the structural framework (both Merton and FPT frameworks). We study the calibration of structural frameworks using a market implied volatility skew. We show that the model implied volatility skew under FPT framework is much more flexible than that under the Merton framework. Moreover, we extend the FPT structural framework to include warrants into the firm's capital structure. Using historical market data, we show the pricing model (for both options and warrants) under FPT framework significantly outperforms the pricing models under Merton framework.

Keywords: Damages; Misrepresentation; Securities Class Actions; Capital Structure; Debt; Warrants; Merton Model; Connection Between Unobservable Firm Value and Observable Stock Price; Structural Models; Pricing; Valuation With Observable Information; Volatility Skew; Warrants Pricing;

Co-Authorship Statement

All three of my papers included in this thesis are co-authored with my Supervisor R. Mark Reesor. I, Xinghua Zhou, am the primary author.

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Chapter 1

Introduction

There are three major classes of models in the credit risk modeling literature. They are the credit scoring models that to predict probability of default pioneered by Altman [1], the structural models first developed by Black and Scholes [5], and Merton [32], and the reduced-form model originated by Jarrow and Turnbull [23], Jarrow and Turnbull [24], and Duffie and Singleton [13]. My thesis focuses on contingent claims valuation of corporate securities using structural models and applications in damages estimation for securities class actions. In this chapter, we give a brief introduction to structural models and their calibration.

1.1 Brief Overview

Structural models assume that complete information of firm assets and liabilities is given and that firm value can be described by a diffusion process. Securities issued by the firm can be viewed as contingent claims written on firm value. The structural model provides an intuitive and unified framework for pricing debt, equity and other securities such as options and warrants. Structural frameworks are also used in estimating default probabilities, credit ratings and loss distributions of credit portfolios. As discussed later, the structural framework can be used by firm management to determine the best mix of securities to issue for financing its operations (e.g., capital structure). Moreover, corporate/investment decisions and the impact of policy changes can be studied under the structural framework.

1.1.1 The Merton Framework

The Merton [32] structural framework is developed based on the following assumptions:

- A.1 There are no transactions costs, taxes, or problems with indivisibilities of assets.
- A.2 There are a sufficient number of investors with comparable wealth levels so that each investor believes that he can buy and sell as much of an asset as he wants at the market price.
- A.3 There exists an exchange market for borrowing and lending at the same rate of interest.
- A.4 Short-sales of all assets, with full use of the proceeds, is allowed.

- A.5 Trading in assets takes place continuously in time.
- A.6 The Modigliani-Miller theorem that the value of the firm is invariant to its capital structure obtains.
- A.7 The term structure is “flat” and known with certainty. That is, the price of a riskless discount bond which promises a payment of one dollar τ years in the future is $P(\tau) = \exp[-r\tau]$ where r is the (instantaneous) riskless rate of interest, the same for all time.
- A.8 The dynamics for the firm value, V , through time can be described by a diffusion-type stochastic process with stochastic differential equation

$$dV_t^F = (\mu V_t^F - C)dt + \sigma^F V_t^F dW_t, \quad (1.1)$$

where μ is the instantaneous expected rate of return on firm asset, C is the total dollar payouts by the firm per unit time to either its shareholders or liabilities-holders if positive; and it is the net dollars received by the firm from new financing if negative, σ^F is the firm-value volatility and W_t is a standard Brownian motion. Here the firm value process is specified under the real-world measure. In what follows, the firm value process is specified under a risk neutral measure.

As mentioned in Merton’s paper, the first four assumptions (A.1 to A.4) are the “perfect market” assumptions (same as that used in deriving the Black-Scholes formula) and can be weakened. The other assumptions (A.5 to A.8) are discussed and modified in the subsequent structural model literature. Moreover, the Merton framework assumed that the capital structure of a firm consists of only debt and equity, where debt is represented by a single zero coupon bond with maturity T and face value F and there is one type of equity that pays no dividends. These assumptions imply that the total dollar payouts is zero (e.g., $C = 0$). Default can only happen at the maturity date T . As a result, the equity value can be represented as a European call option written on firm value V^F with strike F . The equity value is given by

$$EQ_t = V_t^F \Phi(d_1) - e^{-r(T-t)} F \Phi(d_2), \quad (1.2)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function and d_1 and d_2 are given by

$$d_1 = (\log(V_t^F / F) + (r + \frac{1}{2}\sigma^{F^2})(T - t)) / (\sigma^F \sqrt{T - t}) \quad (1.3)$$

and

$$d_2 = d_1 - \sigma^F \sqrt{T - t}, \quad (1.4)$$

respectively. The debt value is simply the difference between the firm value and the equity value, i.e.,

$$D_t = V_t^F - EQ_t. \quad (1.5)$$

The assumption that firm liabilities can be represented by a single zero coupon debt is not realistic. Geske [17] extended the Merton model to consider the firm’s liabilities as a coupon-paying bond that makes discrete coupon payments. The value of a coupon bond that makes n coupon payments is given by a formula containing an n -dimensional multivariate normal integral, as the $(i + 1)$ th coupon is only paid if the i th coupon is paid, $i = 1, \dots, n - 1$. In this case, equity is viewed as a compound option on firm value.

1.1.2 The First Passage Time Framework

Black and Cox [4] introduced safety covenants when valuing risky corporate debt. Safety covenants are contractual provisions which give the bondholders the right to bankrupt or force a reorganization of the firm if it is doing poorly according to some standard. The safety covenants are modeled as a default boundary; if the firm value falls below the default boundary, the bondholders are entitled to force the firm into bankruptcy. While the Merton model only allows firm default on debt maturity date T , the Black-Cox model allows default at any time before maturity (when the firm value first reaches the default boundary). The structural model literature usually refers to this modeling setup as the First Passage Time (FPT) structural framework. Nielsen, Saà-Requejo and Santa-Clara [33], Longstaff and Schwartz [29] and Briys and de Varenne [6] adapt the FPT framework to include a stochastic interest rate in their studies (relaxing Assumption A.7 under Merton's framework).

Under the FPT framework, the equity value is given as the value of a European down-and-out call option written on firm value V^F . A bond holder would suffer no loss provided the firm value V_t^F never reaches the default boundary K prior to maturity T . In the case that a firm's liabilities has only one single zero-coupon bond and one type of non-dividend paying equity, we set the default boundary to be equal to the face value of the debt K^1 . The time- t equity value EQ_t is given by

$$EQ_t = V_t^F \Phi(d_1) - Ke^{-r(T-t)} \Phi(d_2) - V_t^F (K/V_t^F)^{2\lambda} \Phi(d_3) + Ke^{-r(T-t)} (K/V_t^F)^{2\lambda-2} \Phi(d_4), \quad (1.6)$$

where

$$\lambda = (r + \sigma^{F^2}/2)/\sigma^{F^2} \quad (1.7)$$

$$d_1 = (\log(V_t^F/K) + (r + \sigma^{F^2}/2)(T-t))/(\sigma^F \sqrt{T-t}), \quad (1.8)$$

$$d_2 = d_1 - \sigma^F \sqrt{T-t}, \quad (1.9)$$

$$d_3 = (\log(K/V_t^F) + (r + \sigma^{F^2}/2)(T-t))/(\sigma^F \sqrt{T-t}), \quad (1.10)$$

and

$$d_4 = d_3 - \sigma^F \sqrt{T-t}. \quad (1.11)$$

When firm value V^F reaches the default boundary K , the bond holder receives the $1 - w$ times the face value of the bond, where w is the percentage loss given default. The time- t price of T -maturity risky debt with face value one is given by

$$P(V_t^F, \sigma^F, r, w, K, t, T) = e^{-r(T-t)} EQ[1 - wI_{\tau \leq T}] = e^{-r(T-t)} (1 - wQ(\tau \leq T)), \quad (1.12)$$

where τ is the first passage time of the firm value V^F to the boundary K , I_A is the indicator function of the event A and $Q(\tau \leq T)$ is the probability of the event $[\tau \leq T]$ under the risk-

¹The default boundary is typically set below the face value of the debt. This requires a straightforward adjustment to the valuation formula here and in what follows in the thesis.

neutral measure. The time- t conditional distribution of the first passage time, τ , is

$$\begin{aligned} Q(\tau \leq T) = & \Phi\left(\frac{-\log(V^F/K) - r(T-t) + 0.5\sigma^{F^2}(T-t)}{\sigma^F \sqrt{T-t}}\right) \\ & + \exp\left(\frac{-2\log(V^F/K)(r - 0.5\sigma^{F^2})}{\sigma^{F^2}}\right) \\ & \times \Phi\left(\frac{-\log(V^F/K) + r(T-t) - 0.5\sigma^{F^2}(T-t)}{\sigma^F \sqrt{T-t}}\right), \end{aligned} \quad (1.13)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. Notice that Black and Cox [4] assume that the default boundary is time dependent, where the default boundary is defined as $\delta K e^{T-t}$ with $\delta \in [0, 1]$. Our papers, however, assume a constant default boundary following Longstaff and Schwartz [29].

1.1.3 Optimal Capital Structures and Other Extensions

The default boundaries we have discussed so far are exogenous, i.e., the default boundary is assigned without any input from the firm's decision makers. Another popular framework is the endogenous default model, which assumes that the default boundary is determined by equity holders/firm management to maximize equity value. Black and Cox [4] first introduced an endogenous default boundary into the structural literature. Leland [26] introduced bankruptcy costs and tax benefits into the first passage time framework with perpetual debt (Assumption A.6 under the Merton framework is relaxed). Leland argued that debt issuance affects the total value of the firm in two ways. Potential bankruptcy costs from debt issuance reduces the total firm value. On the other hand, tax benefits from interest payment increases the total firm value. Under the perpetual debt framework, it can be shown that bankruptcy cost and tax benefit are functions of firm asset value V_t^F (and that they are not time dependent). Given the firm asset value V_t^F (which follows a log normal process), the total value of the firm $v(V_t^F)$ is

$$v(V_t^F) = V_t^F + TB(V_t^F) - BC(V_t^F), \quad (1.14)$$

where $TB()$ is the expected tax benefit and $BC()$ is the expected bankruptcy cost. Optimal capital structure is discussed in Leland [26] under this modeling set up. Leland and Toft [28] study optimal capital structure under a similar modeling framework by relaxing the perpetual debt assumption. Instead, they assume that debt is continuously rolled over.

As will be discussed later, structural models fail to generate high enough credit spreads compared with those observed in the market for short-dated debt. In order to resolve this issue, jump diffusion processes are introduced to model firm value (Assumption A.8 under the Merton framework is relaxed). Firm value processes with jumps have been studied by Zhou [36], Hilberink and Rogers [20] and Chen and Kou [8].

The structural literature is huge and has many branches. Other topics within the structural literature include (but are not limited to): strategic default, where debt holders and equity holders operate within the bankruptcy code to resolve financial distress. (See Anderson and Sundaresan [2], Anderson, Sundaresan and Tychon [3], Mella-Barral and Perraudin [30] and Broadie, Chernov and Sundaresan [7]); agency costs, when investment policies are chosen

to maximize equity value after (i.e., ex post) debt is in place, e.g., stockholder–bondholder conflicts (See Mello and Parsons [31] and Leland [27]); and dynamic capital structure, where optimal leverage is dynamic instead of static (See Fischer, Heinkel and Zechner [16], and Goldstein, Ju and Leland [19]).

1.2 Calibration and Empirical Evidence

As mentioned in the previous section, structural models are developed under the assumption that complete information of firm assets and liabilities are given. That is, models are constructed based on the firm asset value V_t^F , firm asset volatility σ^F , default boundary K and other market factors such as the risk-free rate term structure. However, firm value V_t^F and firm volatility σ^F are not observable from the market. In most of the cases, detailed information about the liabilities (for example) of a firm is not easy to obtain from financial statements. As a result, implementing and testing structural models are challenging. In this section, we go through three calibration approaches appearing in the structural literature.

The first approach is the traditional approach, which involves solving a non-linear system of equations for firm value and firm volatility. Ronn and Verma [34] and Jones, et al. [25] adapted this calibration method in their studies. The traditional approach leverages information from the stock price S_t and stock volatility σ^S to infer firm value and firm volatility. In general, stock volatility σ^S is estimated by assuming the stock price follows

$$dS_t = \mu^S S_t dt + \sigma^S S_t dW_t, \quad (1.15)$$

where μ^S is the drift term for the log normal stock process. Under the structural framework, the stock price is a contingent claim on firm value V_t^F . Hence the stock price is a function of firm value, i.e.,

$$S_t = S(V_t^F), \quad (1.16)$$

where $S()$ is the stock value function. From the firm value process (equation (1.1)), Itô's Lemma and equation (1.16) give the dynamics of the stock value process, which is

$$dS(V_t^F) = \left(S_t(V_t^F) + (\mu V_t^F - C) S_V(V_t^F) + \frac{1}{2} S_{VV}(V_t^F) \sigma^{F2} V_t^{F2} \right) dt + S_V(V_t^F) \sigma^F V_t^F dW_t, \quad (1.17)$$

where $S_t()$ is the partial derivate of $S()$ with respect to (w.r.t.) t , $S_V()$ is the partial derivate of $S()$ w.r.t. V_t^F , and S_{VV} is the partial second derivative of $S()$ w.r.t. V_t^F . By matching the coefficient of the dW_t term between equations (1.15) and (1.17), we get the relationship between firm volatility and stock volatility, which is

$$\sigma^S S_t = S_V(V_t^F) \sigma^F V_t^F. \quad (1.18)$$

Coupling equation (1.18) above with the stock value equation (1.16), a nonlinear system with unknown variables V_t^F and σ^F is constructed. Given all the other parameters, V_t^F and σ^F can be estimated by solving a non-linear system. Chapter 2 and Chapter 3 of this thesis study the traditional approaches and show that instead of solving a non-linear system simultaneously, the

problem can be transformed into solving two non-linear equations sequentially. This technique improves the accuracy in estimating firm value and firm volatility².

The second approach of structural model calibration is called the Maximum Likelihood Estimation (MLE) approach. From the discussion of the traditional approach, we notice that stock volatility is estimated by assuming the stock price follows a log normal process with constant volatility. Duan [10], [11], Ericsson and Reneby [15], and Duan, et al. [12] argue that this estimation method is misspecified, given that structural models imply that the equity volatility σ^E is a function of V_t^F and σ^F . The MLE approach, first introduced by Duan [10], estimates firm volatility by using the time series of stock prices under a structural model framework. Let $S^{TS} \equiv \{S_i^{ts} : i = 1, 2, \dots, n\}$ be a time series of stock price, where i is the time index, $\mu = r + \lambda^v \sigma^F$ and $C = 0$ in equation (1.1), and $f(\cdot)$ be the conditional density of S_i^{ts} given S_{i-1}^{ts} , the log-likelihood function for vector S^{TS} is

$$L_S(S^{TS}; \sigma^F, \lambda^v) = \sum_{i=2}^n \log f(S_i^{ts} | S_{i-1}^{ts}; \sigma^F, \lambda^v), \quad (1.19)$$

where λ^v is usually referred to be the market price of risk. We know that $\log(V_t^F)$ is normally distributed, and its conditional density function of $g(\cdot)$ is given by

$$g(\log V_i^F | \log V_{i-1}^F; \sigma^F, \lambda^v) = \frac{1}{\sqrt{2\pi s_i^2}} \exp\left(-\frac{(\log V_i^F - m_i)^2}{2s_i^2}\right), \quad (1.20)$$

where

$$m_i = \log V_{i-1}^F + (r + \lambda^v \sigma^F - 0.5\sigma^{F^2})\Delta t, \quad (1.21)$$

$$s_i = \sigma^F \sqrt{\Delta t}, \quad (1.22)$$

and $\Delta t = t_i - t_{i-1}$. If V_t^F is a one-to-one and hence invertible function of S_t , the conditional density $f(\cdot)$ can be written as

$$f(S_i^{ts} | S_{i-1}^{ts}; \sigma^F, \lambda^v) = g(\log V_i^F | \log V_{i-1}^F; \sigma^F, \lambda^v) \Big|_{V_i^F = v_{S_i^{ts}}(\sigma^F)} \times \left(\frac{\partial S_i}{\partial \log V_i^F} \Big|_{V_i^F = v_{S_i^{ts}}(\sigma^F)}\right)^{-1}, \quad (1.23)$$

where $v_{S_i^{ts}}()$ is a function of σ^F for a given stock price S_t . With equation (1.23), firm volatility is estimated by maximizing the log-likelihood function in equation (1.19). Given the firm volatility σ^F , firm value V_t^F is calculated as $V_t^F = v_{S_t^{ts}}(\sigma^F)$. Using simulation, Ericsson and Reneby[15] show that the MLE method proposed by Duan[10] does a better job than the traditional method in estimating the firm value V_t^F and firm volatility σ^F . The MLE approach in the literature is studied using the Merton model. Chapter 3 extends the MLE approach into the First Passage Time framework by showing that the firm value V_t^F is a one-to-one function of S_t .

The third approach calibrates the structural framework using information from the implied volatility skew from the equity options market. In the Black-Scholes option pricing framework,

²Note that by equating the drift terms in equations (1.15) and (1.17) a Black-Scholes-type PDE is derived for the stock price, namely, $\mu^S S_t = S_t(V_t^F) + (\mu V_t^F - C)S_V(V_t^F) + \frac{1}{2}S_{VV}\sigma^{F^2}V_t^{F^2}$.

the stock price process is assumed to follow geometric Brownian motion as in equation (1.15). However, under the structural framework, the stock price is a call option on firm value V_t^F and hence an equity call option is a compound option on the firm value V_t^F . As shown by equation (1.17), the stock value process follows a diffusion process with instantaneous volatility $S_V(V_t^F)\sigma^F V_t^F$ under the structural framework. In other words, the stock volatility under structural models is stochastic when the firm value volatility is constant. As a result, the structural model implies a volatility skew in option pricing. The idea of the third approach is to fit the model implied volatility skew to the market implied volatility skew. Advantages of the option calibration approach is that it is a point in time calibration and that in addition to firm value and firm volatility, even the default boundary K or the leverage ratio $L \equiv K/V_t^F$ can be inferred from the market. This approach is also forward-looking as the options maturity dates are in the future. Hull, Nelken and White [22] and Geske, Subrahmanyam, Avandhar and Zhou [18] introduces this calibration approach for the Merton model and concluded that the options market contain insightful information of firm leverage. Detailed discussion of the option calibration approach under both Merton and FPT frameworks is given in Chapter 4.

In the rest of this section, we briefly discuss some empirical studies in the structural model literature. Eom, Helwege and Huang [14] test the performance of five structural models in bond pricing using 182 bond prices from firms with simple capital structures during the period 1986 - 1997. They find that on average structural models under the Merton framework (Merton [32] and Geske [17]) failed to generate high enough corporate bond spreads as compared to those observed in markets, while structural models under FPT framework (Longstaff and Schwartz [29], Leland and Toft [28] and Collin-Dufresne and Goldstein [9]) overestimate the observed spreads. Huang and Huang [21] find that credit risk accounts for only a small fraction of the observed corporate-Treasury yield spreads for investment grade bonds of all maturities and that it accounts for a much higher fraction of yield spreads for non-investment grade bonds. Moreover, even for models with jumps (Zhou [36]), the structural model is unable to generate high enough yield spreads for short maturity bonds when the structural model is calibrated to the historical default probability. The failure of structural models in explaining both yield spreads and default probability is referred to as the credit spread puzzle. One explanation of the credit spread puzzle is that some portion of corporate-Treasury yield spreads is due to some non-credit related factors such as liquidity. However, even though structural models provide poor predictions of bond prices, Schaefer and Strebulaev [35] find that they provide quite accurate predictions of the sensitivity of corporate bond returns to changes in the value of equity (hedge ratios).

1.3 Structure of Thesis

This thesis studies structural models with a focus on its applications and calibration. Under the Merton structural framework, Chapter 2 discusses the impact of misrepresentation on corporate issued securities such as debt, warrants and equity. Leveraging the Merton framework, we investigate various capital structures and show that misrepresentation has significant impact on the value of all components in the capital structure. Using a FPT structural model and leveraging the relationship between equity and firm value, Chapter 3 proposes a methodology that allows for debt damages assessments consistent with standard methods for assessing eq-

uity damages. In addition to our main findings, we explicitly discuss bankruptcy costs in the FPT model. Chapter 4 studies option pricing and warrant pricing under both Merton and FPT structural frameworks. By extending the FPT structural framework to include warrants into a firm's capital structure we study leverage and dilution effects in pricing warrants. Moreover, we study the calibration of structural frameworks using the market-implied volatility skew.

Bibliography

- [1] Edward I. Altman. Financial ratios, discriminant analysis and the prediction of corporate bankruptcy. *The Journal of Finance*, 23(4):589–609, 1968.
- [2] Ronald W. Anderson and Suresh Sundaresan. Design and valuation of debt contracts. *The Review of Financial Studies*, 9(1):37–68, 1996.
- [3] Ronald W. Anderson, Suresh Sundaresan, and Pierre Tychon. Strategic analysis of contingent claims. *European Economic Review*, 40(3):871 – 881, 1996. Papers and Proceedings of the Tenth Annual Congress of the European Economic Association.
- [4] Fischer Black and John C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367, 1976.
- [5] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, pages 637–654, 1973.
- [6] Eric Briys and Franois de Varenne. Valuing risky fixed rate debt: An extension. *The Journal of Financial and Quantitative Analysis*, 32(2):239–248, 1997.
- [7] Mark Broadie, Mikhail Chernov, and Suresh Sundaresan. Optimal debt and equity values in the presence of chapter 7 and chapter 11. *The Journal of Finance*, 62(3):1341–1377, 2007.
- [8] Nan Chen and S. G. Kou. Credit spreads, optimal capital structure, and implied volatility with endogenous default and jump risk. *Mathematical Finance*, 19(3):343–378, 2009.
- [9] Pierre Collin-Dufresne and Robert S. Goldstein. Do credit spreads reflect stationary leverage ratios? *The Journal of Finance*, 56(5):1929–1957, 2001.
- [10] Jin-Chuan Duan. Maximum likelihood estimation using price data of the derivative contract. *Mathematical Finance*, 4(2):155–167, 1994.
- [11] Jin-Chuan Duan. Correction: Maximum likelihood estimation using price data of the derivative contract (mathematical finance 1994, 4/2, 155167). *Mathematical Finance*, 10(4):461–462, 2000.
- [12] Jin-Chuan Duan, Genevieve Gauthier, and Jean-Guy Simonato. On the equivalence of the KMV and maximum likelihood methods for structural credit risk models. Working Paper, 2005.

- [13] Darrell Duffie and Kenneth J. Singleton. Modeling term structures of defaultable bonds. *The Review of Financial Studies*, 12(4):687–720, 1999.
- [14] Young Ho Eom, Jean Helwege, and Jing-Zhi Huang. Structural models of corporate bond pricing: An empirical analysis. *Review of Financial Studies*, 17(2):499–544, 2004.
- [15] Jan Ericsson and Joel Reneby. Estimating structural bond pricing models*. *The Journal of Business*, 78(2):707–735, 2005.
- [16] Edwin O. Fischer, Robert Heinkel, and Josef Zechner. Dynamic capital structure choice: Theory and tests. *The Journal of Finance*, 44(1):19–40, 1989.
- [17] Robert Geske. The valuation of corporate liabilities as compound options. *The Journal of Financial and Quantitative Analysis*, 12(4):541–552, 1977.
- [18] Robert Geske, Avanidhar Subrahmanyam, and Yi Zhou. Capital structure effects on the prices of equity call options. *Journal of Financial Economics*, 121(2):231 – 253, 2016.
- [19] Robert Goldstein, Nengjiu Ju, and Hayne Leland. An EBIT-based model of dynamic capital structure. *The Journal of Business*, 74(4):483–512, 2001.
- [20] Bianca Hilberink and Leonard CG Rogers. Optimal capital structure and endogenous default. *Finance and Stochastics*, 6(2):237–263, 2002.
- [21] Jing-Zhi Huang and Ming Huang. How much of the corporate-treasury yield spread is due to credit risk? *Review of Asset Pricing Studies*, 2(2):153–202, 2012.
- [22] John Hull, Izzy Nelken, and Alan White. Mertons model, credit risk, and volatility skews. *Journal of Credit Risk*, 1(1), 2004.
- [23] Robert A. Jarrow and Stuart M. Turnbull. Credit risk: Drawing the analogy. *Risk Magazine*, 5(9):63–70, 1992.
- [24] Robert A. Jarrow and Stuart M. Turnbull. Pricing derivatives on financial securities subject to credit risk. *The Journal of Finance*, 50(1):53–85, 1995.
- [25] E. Philip Jones, Scott P. Mason, and Eric Rosenfeld. Contingent claims analysis of corporate capital structures: an empirical investigation. *The Journal of Finance*, 39(3):611–625, 1984.
- [26] Hayne E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *The Journal of Finance*, 49(4):1213–1252, 1994.
- [27] Hayne E. Leland. Agency costs, risk management, and capital structure. *The Journal of Finance*, 53(4):1213–1243, 1998.
- [28] Hayne E. Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987–1019, 1996.

- [29] Francis A. Longstaff and Eduardo S. Schwartz. A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3):789–819, 1995.
- [30] Pierre Mella-Barral and William Perraudin. Strategic debt service. *The Journal of Finance*, 52(2):531–556, 1997.
- [31] Antonio S. Mello and John E. Parsons. Measuring the agency cost of debt. *The Journal of Finance*, 47(5):1887–1904, 1992.
- [32] Robert C Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.
- [33] Lars Tyge Nielsen, Jesus Saà-Requejo, and Pedro Santa-Clara. *Default risk and interest rate risk: The term structure of default spreads*. INSEAD, 1993.
- [34] Ehud I. Ronn and Avinash K. Verma. Pricing risk-adjusted deposit insurance: An option-based model. *The Journal of Finance*, 41(4):871–896, 1986.
- [35] Stephen M. Schaefer and Ilya A. Strebulaev. Structural models of credit risk are useful: Evidence from hedge ratios on corporate bonds. *Journal of Financial Economics*, 90(1):1 – 19, 2008.
- [36] Chunsheng Zhou. The term structure of credit spreads with jump risk. *Journal of Banking & Finance*, 25(11):2015–2040, 2001.

Chapter 2

Misrepresentation and Capital Structure: A Modified Merton Framework

2.1 Introduction

A variety of illegal activities such as Ponzi schemes, insider trading, market manipulation and misrepresentation can lead to regulatory enforcement proceedings, criminal prosecution and/or securities class action lawsuits. This chapter concerns the assessment of damages in secondary market securities class actions as a result of misrepresentation by a firm (including its directors and officers). Examples of misrepresentation include the overstatement of earnings, the failure to properly disclose the risks of potential liabilities, and accounting irregularities. Typically, misrepresentation results in an overstatement of firm value. For firms whose shares trade in an efficient secondary market, the share price of the firm quickly drops when the misrepresentation is revealed. Using well-established econometric methods, the share price drop is used to assess potential damages to investors who transacted in the share during the class period — the time between the start of the misrepresentation and its revelation.¹ In securities class actions, the start date of fraud and the fraud disclosure date (and hence the class period) are determined by the court. In the following study, we assume that the class period is given.

Firms use many vehicles to finance their operations including common and preferred shares, warrants, various debt instruments and employee stock options. The instruments used determine a firm's capital structure and the firm value equals the total value of the component instruments in the capital structure. Fraud affects the value of the entire firm, not just the value of equity. Therefore, assessing damages only due to shareholders can lead to a significant understatement of the losses incurred by investors across all of the firm's issued securities. For example, the recent case of Sino-Forest in which equityholders were completely wiped out and control of the firm's residual value reverted to bondholders, clearly shows that fraud can induce losses to debtholders [4]. In most instances of misrepresentation, equityholders are not completely wiped out, but the misrepresentation still affects the debt value.

In this chapter we use a modified Merton framework to measure the impact of misrepresentation on the value of other components (e.g., debt, warrants) of a firm's capital structure. Using

¹In this paper we use the terms misrepresentation and fraud interchangeably, although we recognize there are important legal distinctions between them.

a relationship between equity and firm value we show how observable equity information can be used to determine firm value and hence the value of other securities in the capital structure. Thus the effect of misrepresentation on firm value and the capital structure constituents can be measured from the observable drop in share price. This leads to total damages assessment consistent with the standard method for assessing damages to equity investors. Furthermore, trades involving corporate bonds and warrants happen much less frequently than for common shares and hence corporate debt and warrant markets are typically less efficient than equity markets. For example, during the class period there could either be i) no trades involving that firm's debt (i.e., no information about bond value change); or ii) very few trades in an inefficient market (i.e., potentially unreliable information about bond value change). In such situations, the modelling framework presented here allows one to compute bond value changes consistent with the observed share price change. In the structural model literature, the effects of accounting uncertainty has been discussed in [8], which is relevant to the effect of misrepresentation (which is defined as the difference in security values with and without the existence of fraud) discussed in this thesis. But our works see this issue from a different angle and with a different scope. First, [8] studies the effects of potential accounting noise in pricing risky debt in the future, while our works focus on the effect of misrepresentation which has already happened in some previous period (i.e., the class period). Second, the modeling scopes are different between these two models. The effects of misrepresentation discussed in this thesis are not limited to accounting frauds and include the fraud impacts on the values of various securities including equity, warrants, and debt. Third, under the modeling framework in [8], the effects of accounting fraud can be viewed as the firm's management intentionally creating noise in the accounting report of assets. In this thesis, we adapt the classic structural framework for the reason of simplicity and clarity. Also note that this paper concerns only the impact of fraud on the value of firm-issued securities, not on the value of third-party issued securities, such as exchange-traded single-name equity options.

We investigate various capital structures and show that misrepresentation has a significant impact on the value of all components in the capital structure. We find that the misrepresentation impact on debt value depends on firm leverage and debt seniority and not on the warrant dilution factor. Generally, the debt for higher-leverage firms is more sensitive to the misrepresentation impact than for lower-leverage firms and junior debt is more affected by fraud than senior debt. The impact on warrant value is determined by warrant moneyness (stock price), with the dilution factor having no effect.

Our findings have important consequences for damages assessment and allocation of settlement awards in securities class actions. Additionally in some jurisdictions (e.g., Ontario) damages awarded are net of any hedge or risk-limitation transaction. Since corporate securities such as bonds, stocks and warrants are often held in portfolios for hedging purposes, measuring the effect of misrepresentation on all of the firm's issuances is essential to accurately computed damages awards.

In addition to our main findings, we provide ancillary contributions to the warrant valuation and capital structure literature. These contributions extend the work in Crouhy and Galai's [7] framework to a capital structure that includes both junior and senior debt. Additionally, we broaden the connection between observable stock price and its volatility with unobservable firm value and its volatility discussed in Ukhov[30] and Abinzano and Navas[1] to other capital structures not previously considered.

The paper is organised as follows. The rest of Section 2.1 discusses a standard econometric method for assessing the impact of fraud on share price and also introduces the “fixed ratio change” model. Section 2.2 discusses the modified Merton framework, gives examples of several simple capital structures and provides valuation formulae for each. Section 2.3 gives the connection between the unobservable firm value and the observable share price, critical to measuring the impact of fraud on the other securities in the capital structure. In Section 2.4 we investigate the effect of fraud on various capital structures and discuss some key findings. Section 2.5 illustrates the damages calculation methodology with a case study. A summary and concluding discussion is given in Section 2.6.

2.1.1 The Value Line and Fixed Ratio Change Model

An efficient market² is one in which publicly-available information and signals are quickly evaluated and reflected in market prices. Stock markets such as NYSE and TSX are considered efficient markets. There are many different ways to measure market efficiency available in the literature [14],[3],[12], [27]. Both firm-specific (idiosyncratic) and general market (systematic) information affect a firm’s stock price. Under this presumption the stock price is viewed as a function of both pieces of information.

A two-factor linear model is the basic financial econometric model used to estimate a security’s true value and from which damages estimates are computed. This model specifies the security’s expected return as a linear function of the return on the whole market and the return on the industry sector. Specifically,

$$R_t = \alpha + \beta_1 R_{Mt} + \beta_2 R_{It} + e_t, \quad (2.1)$$

where R_t , R_{Mt} , and R_{It} are the time- t stock, market, and industry returns, respectively, and e_t is the residual value (assumed to be independent random variables with mean zero and constant variance). Using Equation 2.1 the expected return of the stock at time t is

$$E[R_t] = \alpha + \beta_1 R_{Mt} + \beta_2 R_{It}. \quad (2.2)$$

Regression analysis with historical data is performed to estimate α , β_1 , and β_2 . When estimating the econometric model, the sample period should be chosen such that fraudulent information does not affect the normal relationship between the security, the market, and the industry. For details of choosing the sample period, please refer to [5]. Using this model, a value line representing the true stock value (i.e., the path the stock price would have followed in the absence of misrepresentation) can easily be constructed [5]. This method constructs a series of daily returns R_{Ct} in the following way

- if no fraud-related information is disclosed, set the return equal to the actual return on the security;
- if fraud-related information is disclosed or leaked into the market, set the return for these days equal to the expected return calculated using Equation 2.2 and the estimates for α, β_1, β_2 .

²Economists typically refer to three forms of financial market efficiency — weak, semi-strong and strong. The reader is referred to [13] for further discussion of the Efficient Market Hypothesis. Discussion about market efficiency can be found in Malkiel, Burton G., and Eugene F. Fama’s “Efficient capital markets: A review of theory and empirical work.”, *The Journal of Finance* 25.2 (1970): 383-417.

There are other factor models for returns such as the CAPM [11], [20], [24], [26] and three-factor Fama-French model [15]. We chose to use a two-factor linear model following [5]. The stock value line is an input to the methodology developed in this thesis and the methodology does not change according to model choice for returns. This series of daily returns is then used to construct the value line by

$$\tilde{S}_{t-1} = \tilde{S}_t / (1 + R_{Ct-1}), \quad (2.3)$$

where \tilde{S}_t is the time- t stock value.

Much work has been done on the methodology for estimating damages in securities fraud cases. A non-exhaustive list is [23], [17], [16], [5], [9], [29] and [28] who provide variants on the methodology given here. What is missing in these works, however, is the impact of fraud on the value of other securities in the capital structure. In this paper our proposed method assesses the impact of fraud on the value of other securities, given the easily measured and observed impact on the value of share price.

Suppose that τ_b is the start date of the fraud and τ_e is the date the misrepresentation is revealed, so that $[\tau_b, \tau_e]$ defines the class period. The fixed ratio change model assumes that misrepresentation has a proportional effect on the stock price, i.e.,

$$\tilde{S}_t = \delta S_t, \quad (2.4)$$

for $t \in [\tau_b, \tau_e]$, where δ is the fixed ratio change during the class period. Typically, δ is determined by the size of the stock price movement controlled for changes in the market and industry on the date the fraud is revealed. This is used with the model and method discussed above to construct the stock's value line. An alternative approach to the fixed ratio change model is the fixed dollar amount change model, not used in this paper.

2.2 Modified Merton Framework, Capital Structure and the Valuation of Equity, Debt, and Warrants

In this section we review some well-known results that give the value of equity, debt, and warrants for firms across a variety of simple capital structures. We follow the Merton [25] framework and assume the firm value V^F has the following dynamics³

$$dV^F = rV^F dt + \sigma^F V^F dB, \quad (2.5)$$

where r is the risk-free rate, σ^F is the firm-value volatility and B is a risk-neutral standard Brownian motion. There is a future time T at which the firm is liquidated and the proceeds disbursed to the securityholders. For firms financed partially with debt, the debt is zero-coupon and it all matures at time T . This includes the case for which part of the debt is subordinated. For firms with warrants, we follow the set up in Crouhy and Galai [7] in which the warrants are European style and expire at time T_w where $T_w \leq T$. We use E^Q , D , and W to denote the value of equity, debt and warrants, respectively, and a subscript- t on these (and other) quantities denotes their time- t values.

³Here we assume that the firm pays no dividend. However, the modeling framework discussed in this thesis can be easily extended to include continuously-paid dividends q by replacing r by $r - q$ in the formulas.

For many reasons this modelling framework is a coarse simplification of the real-world phenomenon. However, this framework is well known and it forms the basis for many of the popular models used for valuing credit-risky bonds and in determining optimal capital structures. As such it is a natural place to begin to link the observed fraud-induced share price change to the value of other securities in the capital structure.

2.2.1 Debt and Equity Capital Structure

Consider a firm financed with only a single type of (zero-coupon) debt and common shares. At time T the debtholders get paid $\min(V_T^F, F)$ where F is the face value of the debt. The equity holders receive $\max(V_T^F - F, 0)$, the payoff of a call option written on firm value struck at the face value of the debt. Thus, the firm's time- t equity value is

$$\begin{aligned} E_t^Q &= C(V_t^F, F, \sigma^F, T - t) \\ &= V_t^F \Phi(d_1) - e^{-r(T-t)} F \Phi(d_2), \end{aligned} \quad (2.6)$$

where $C(V^F, F, \sigma^F, T - t)$ is the Black-Scholes formula for a European call option on V^F , with strike F , volatility σ^F and time to maturity $T - t$, $\Phi(\cdot)$ is the standard normal cumulative distribution function and d_1 and d_2 are given by

$$d_1 = (\log(V_t^F / F) + (r + \frac{1}{2}\sigma^{F^2})(T - t)) / (\sigma^F \sqrt{T - t}) \quad (2.7)$$

and

$$d_2 = d_1 - \sigma^F \sqrt{T - t}, \quad (2.8)$$

respectively. The firm value is the sum of the value of equity and debt. So the time- t debt value can be written as the difference between firm and equity values,

$$D_t = V_t^F - E_t^Q. \quad (2.9)$$

Given the number of outstanding shares, N , of the firm, the stock price S_t is just the equity value divided by N , which is

$$S_t = C(V_t^F, F, \sigma^F, T - t) / N. \quad (2.10)$$

2.2.2 Junior and Senior Debt and Equity Capital Structure

Here we add zero-coupon subordinated debt to the capital structure from the previous subsection. Subordinated debt has a lower priority than senior debt at the time of liquidation. Let D^S and D^J denote the senior and subordinated debt values, respectively. Assume that both types of debt are zero-coupon and have the same maturity date T . The time- T payoffs for senior debt, subordinated debt, and equity on date T are given in Table (2.1). F^S and F^J are the face value of senior and junior debt, respectively. We can evaluate the time- t prices based on the payoffs given in Table (2.1). The equity payoff at time T is the same as a European call option on V^F with strike $F^S + F^J$. Thus the time- t firm equity value can be written as

$$E_t^Q = C(V_t^F, F^J + F^S, \sigma^F, T - t). \quad (2.11)$$

Table 2.1: Time- T Payoff

	Pay-off at maturity date T
E^Q	$\max\{V_T^F - F^S - F^J, 0\}$
D^J	$\min\{\max\{V_T^F - F^S, 0\}, F^J\}$
D^S	$\min\{V_T^F, F^S\}$

It is easily shown that the payoff of the sum of junior debt and equity is the same as a European call on V^F with strike price F^S , i.e.,

$$D_t^J + E_t^Q = C(V_t^F, F^S, \sigma^F, T - t). \quad (2.12)$$

Hence the value of senior debt D_t^S is

$$\begin{aligned} D_t^S &= V_t^F - (E_t^Q + D_t^J) \\ &= V_t^F - C(V_t^F, F^S, \sigma^F, T - t). \end{aligned} \quad (2.13)$$

The junior debt position can also be expressed as a call bull spread, which has value

$$\begin{aligned} D_t^J &= V_t^F - D_t^S - E_t^Q \\ &= C(V_t^F, F^S, \sigma^F, T - t) - C(V_t^F, F^S + F^J, \sigma^F, T - t). \end{aligned} \quad (2.14)$$

The stock price of the firm is

$$S_t = C(V_t^F, F^J + F^S, \sigma^F, T - t)/N. \quad (2.15)$$

2.2.3 Warrants and Common Shares Capital Structure

In this section, we consider a pure equity firm which issues M warrants and N common shares. A warrant is similar to a call option and gives its holder the right but not the obligation to buy one share of stock at strike price K on maturity date T_w . When a warrant is exercised, the firm issues a new common share and sells it to the warrant holder for K dollars. The newly-issued share dilutes the interest of existing shareholders and the cash infusion of K dollars increases firm value. Even though most warrants are American style, here we consider European style warrants for simplicity — avoiding the complication of dealing with early-exercise. This model was first proposed by Galai and Schneller [18] and has been studied by many authors including Lauterbach and Schultz [22], Crouhy and Galai [6], and Hauser and Lauterbach [19].

When a warrant is exercised, the value of the firm increases by the amount of cash from exercising (K dollars for each warrant exercised). Thus the stock price at T_w immediately after M warrants are exercised is

$$S_{T_w}^W = (V_{T_w}^{F*} + MK)/(M + N), \quad (2.16)$$

where V_t^{F*} denotes the time- t value of an otherwise identical firm which financed itself entirely with common shares⁴, i.e. $V_t^{F*} = V_t^F$ for all $t \leq T_w$. The time- T_w warrant payoff can be written

⁴The benchmark firm is an otherwise identical firm that has only equity in its capital structure and is not financed with warrants. This facilitates valuation and has also been used in [18] and [6].

as

$$\begin{aligned} W_{T_w} &= \max\{(V_{T_w}^{F*} + MK)/(M + N) - K, 0\} \\ &= \frac{1}{M + N} \max\{V_{T_w}^{F*} - NK, 0\}. \end{aligned} \quad (2.17)$$

It is similar to a fractional payoff of a European call option written on V^{F*} with strike price NK . For this capital structure it is easy to show from Equation (2.17) that the exercise trigger of the warrant is the same as that for a European call option, namely exercise if $S_{T_w} \geq K$. Thus we can write the price of the warrant as

$$W_t = \frac{1}{M + N} C(V_t^{F*}, NK, \sigma^F, T_w - t). \quad (2.18)$$

Since the firm value is the sum of value of common shares and warrants, the stock price incorporating the dilution effect from M warrants is

$$S_t = (V_t^F - MW_t)/N. \quad (2.19)$$

2.2.4 Debt, Warrants, and Common Shares Capital Structure

In this section, we introduce Crouhy and Galai's [7] model in which the firm is financed with debt, equity and warrants. The number of warrants is M and each warrant promises the holder the right to purchase one common share at strike price K on maturity date T_w . We only consider the case $T_w \leq T$. A benchmark firm, whose value is denoted by V^{F*} , is an identical firm financed entirely by common shares, i.e., the value of the two firms are the same except on the date when warrants are exercised. So $V_t^F = V_t^{F*}$ for $t \leq T_w$. The firm value for $t \in (T_w, T)$ depends on whether the warrants are exercised, i.e.,

$$V_t^F = \begin{cases} (1 + MK/V_{T_w}^{F*})V_t^{F*} & \text{if warrants exercised} \\ V_t^{F*} & \text{otherwise} \end{cases}.$$

The cash from exercising the M warrants are assumed to be reinvested in the firm, increasing its size proportionally. After the maturity date T_w , the capital structure of the firm consists only of debt and equity, no matter whether the warrants are exercised or not. The pricing formula for the debt and equity capital structure is discussed in Sections 2.2.1 and 2.2.2. The time- T payoffs of the capital structure components are given in Table (2.2).

Table 2.2: Time- T Payoffs

	Warrants exercised	Warrants not exercised
E_T^Q	$\max\{(1 + MK/V_{T_w}^{F*})V_T^{F*} - F, 0\}$	$\max\{V_T^{F*} - F, 0\}$
D_T	$\min\{(1 + MK/V_{T_w}^{F*})V_T^{F*}, F\}$	$\min\{V_T^{F*}, F\}$
W_T	0	0

From Table (2.2) we see that the prices of debt and common shares depend on warrant exercise. Moreover, for this capital structure the exercise criteria for warrants will be defined

based on the firm value instead of the firm's stock price, the exercise criteria for a pure equity firm. In other words, even if $S_{T_W} > K$ on maturity date T_W , it may not be optimal to exercise the warrants because the cash from warrant exercise will partially flow to the debt value and hence it is possible to cause the stock price to drop below the strike price right after warrant exercise.

To show this fact we examine the stock price immediately after warrant exercise. Let $S_{T_W}^W$ be the stock price right after warrant exercise and $S_{T_W}^{NW}$ be the stock price without exercise. Then the time- T_W value of common shares is given by

$$S_{T_W} = \begin{cases} \frac{V_{T_W}^{F*} - D_{T_W}^W + MK}{N + M} \equiv S_{T_W}^W & \text{if warrants exercised} \\ \frac{V_{T_W}^{F*} - D_{T_W}^{NW}}{N} \equiv S_{T_W}^{NW} & \text{if warrants not exercised} \end{cases},$$

where $D_{T_W}^W$ and $D_{T_W}^{NW}$ denote values of debt with and without warrant exercise, respectively. It can be seen that both $S_{T_W}^W$ and $S_{T_W}^{NW}$ are functions of firm value, $V_{T_W}^{F*}$, given that $D_{T_W}^W$ and $D_{T_W}^{NW}$ are also functions of $V_{T_W}^{F*}$. To determine the warrant exercise criteria, we need to find the threshold value of the firm, \bar{V}^F , such that $S_{T_W}^W(\bar{V}^F) = K$. If $V_{T_W}^F \geq \bar{V}^F$ the warrant holders should exercise their warrants, otherwise exercising the warrants would generate a negative payoff. Crouhy and Galai [7] show that even if the stock price is higher than the strike K , it might not be optimal for warrant holders to exercise their warrants, i.e., there exists V^F such that $S_{T_W}^{NW}(V^F) > K$ but $S_{T_W}^W(V^F) < K$. This implies the stock price is no longer a correct signal for exercising warrants.

The firm value $V_{T_W}^F$ at time T_W would be $V_{T_W}^{F*}$ if warrants were not exercised and $V_{T_W}^{F*} + MK$ if warrants were exercised. As discussed before, we assume that the return of the firm value V_t^F is proportional to the benchmark firm's value V_t^{F*} for $t \in (T_W, T]$. Since the values of debt and equity are based on firm value V_t^F , the values right after the maturity date T_W can be determined and are given in Table (2.3).

Table 2.3: Value of debt and equity immediately after T_W .

	Warrants exercised ($V_{T_W}^{F*} \geq \bar{V}^F$)	Warrants not exercised ($V_{T_W}^{F*} < \bar{V}^F$)
S_{T_W}	$C(V_{T_W}^{F*} + MK, F, \sigma^F, T - T_W)/(M + N)$	$C(V_{T_W}^{F*}, F, \sigma^F, T - T_W)/N$
D_{T_W}	$V_{T_W}^{F*} + MK - C(V_{T_W}^{F*} + MK, F, \sigma^F, T - T_W)$	$V_{T_W}^{F*} - C(V_{T_W}^{F*}, F, \sigma^F, T - T_W)$

Based on Table (2.3), we can compute the value of debt and equity at time $t < T_W$ by the expected discounted value of the time- T_W payoff under the risk-neutral measure. The benchmark firm value V_t^{F*} dynamics are

$$dV_t^{F*} = rV_t^{F*}dt + \sigma^F V_t^{F*}dB_t, \quad (2.20)$$

where r is the risk-free rate, assumed to be constant, and B_t is a standard Brownian motion under the risk neutral measure. This assumption is equivalent to

$$V_{T_W}^{F*} = V_t^{F*} \exp\{(r - \frac{1}{2}\sigma^{F2})(T_W - t) + \sigma^F \sqrt{T_W - t}Z\}, \quad (2.21)$$

where Z is a standard normal random variable. Given V_t^{F*} , r , and σ^F , $V_{T_W}^{F*}$ is a one to one function of Z . Hence we can find the threshold value \bar{Z} corresponding to \bar{V}^F , i.e.,

$$\begin{aligned}\bar{V}^F &= V_t^{F*} \exp\left\{\left(r - \frac{1}{2}\sigma^{F^2}\right)(T_W - t) + \sigma^F \sqrt{T_W - t}\bar{Z}\right\} \\ \Leftrightarrow \bar{Z} &= \frac{\log\left(\frac{\bar{V}^F}{V_t^{F*}}\right) - \left(r - \frac{1}{2}\sigma^{F^2}\right)(T_W - t)}{\sigma^F \sqrt{T_W - t}},\end{aligned}\quad (2.22)$$

since V_t^{F*} is an increasing function of Z . If $Z \geq \bar{Z}$, warrants should be exercised, otherwise warrants should not be exercised.

Given the firm value V_t^{F*} at time $t < T_W$, the probability distribution function of $V_{T_W}^{F*}$ can be found (it is lognormal), and hence the value of stock, debt and warrants. The stock price at time t is

$$\begin{aligned}S_t &= e^{-r(T_W-t)} \left\{ E\left[\frac{C(V_{T_W}^{F*} + MK, F, \sigma^F, T - T_W)}{M + N}\right] 1_{(V_{T_W}^{F*} \geq \bar{V}^F)} \right. \\ &\quad \left. + E\left[\frac{C(V_{T_W}^{F*}, F, \sigma^F, T - T_W)}{N}\right] 1_{(V_{T_W}^{F*} < \bar{V}^F)} \right\} \\ &= e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{\bar{Z}}^{+\infty} \frac{C(V_{T_W}^{F*} + MK, F, \sigma^F, T - T_W)}{M + N} \exp\left\{\frac{-x^2}{2(T_W-t)}\right\} dx \\ &\quad + e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{-\infty}^{\bar{Z}} \frac{C(V_{T_W}^{F*}, F, \sigma^F, T - T_W)}{N} \exp\left\{\frac{-x^2}{2(T_W-t)}\right\} dx.\end{aligned}\quad (2.23)$$

Similarly, the value of debt can be obtained as

$$\begin{aligned}D_t &= e^{-r(T_W-t)} \left\{ E[V_{T_W}^{F*} + MK - C(V_{T_W}^{F*} + MK, F, \sigma^F, T - T_W)] 1_{(V_{T_W}^{F*} \geq \bar{V}^F)} \right. \\ &\quad \left. + E[V_{T_W}^{F*} - C(V_{T_W}^{F*}, F, \sigma^F, T - T_W)] 1_{(V_{T_W}^{F*} < \bar{V}^F)} \right\} \\ &= e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{\bar{Z}}^{+\infty} (V_{T_W}^{F*} + MK - C(V_{T_W}^{F*} + MK, F, \sigma^F, T - T_W)) \\ &\quad \times \exp\left\{\frac{-x^2}{2(T_W-t)}\right\} dx \\ &\quad + e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{-\infty}^{\bar{Z}} (V_{T_W}^{F*} - C(V_{T_W}^{F*}, F, \sigma^F, T - T_W)) \exp\left\{\frac{-x^2}{2(T_W-t)}\right\} dx.\end{aligned}\quad (2.24)$$

The time- t warrant price is

$$\begin{aligned}W_t &= e^{-r(T_W-t)} E\left[\frac{C(V_{T_W}^{F*} + MK, F, \sigma^F, T - T_W)}{M + N} - K\right] 1_{(V_{T_W}^{F*} \geq \bar{V}^F)} \\ &= \frac{e^{-r(T_W-t)}}{\sqrt{2\pi(T_W-t)}} \int_{\bar{Z}}^{+\infty} \left(\frac{C(V_{T_W}^{F*} + MK, F, \sigma^F, T - T_W)}{M + N} - K\right) \exp\left\{\frac{-x^2}{2(T_W-t)}\right\} dx\end{aligned}\quad (2.25)$$

Since firm value is the sum of the values of debt, warrant and equity the warrant price can be computed as

$$W_t = (V_t^{F*} - NS_t - D_t)/M, \quad (2.26)$$

where S_t and D_t are computed using Equations (2.23) and (2.24), respectively.

The valuation formulas contain numerical integrations, where ∞ and $-\infty$ are replaced by 10 and -10 , respectively, in the implementations in this thesis. Under this distribution, there is almost zero probability that the random variable's magnitude will be greater than 10, hence justifying replacing infinity limit with finite limit in the definite integral.

2.2.5 Junior and Senior Debt, Warrants, and Common Shares Capital Structure

In this section we extend Crouhy and Galai's [7] framework to include two different kinds of debt — senior debt and subordinated debt. The capital structure of the firm includes junior and senior debt, warrants, and common shares and hence the firm value is given by

$$V_t^F = D_t^J + D_t^S + W_t + E_t^Q, \quad (2.27)$$

where E_t^Q is the value of all outstanding common shares. As in the previous section, we use V_t^{F*} to denote the benchmark firm which is financed entirely by common shares. We have $V_t^{F*} = V_t^F$ for $t < T_w$ and

$$V_t^F = \begin{cases} (1 + MK/V_{T_w}^{F*})V_t^{F*} & \text{if warrants exercised} \\ V_t^{F*} & \text{otherwise} \end{cases},$$

for $t \in (T_w, T)$. As in Crouhy and Galai [7], we assume that the cash from exercising the warrants is reinvested in the firm, increasing firm value proportionally. We are interested in pricing the equity, debt and warrants before the maturity date T_w . After the warrant maturity date T_w the capital structure of the firm consists only of debt and equity, whose prices are given in Section 2.2.2. The time- T payoff of each capital structure component is given in Table (2.4).

Table 2.4: Time- T Payoffs

	Warrants exercised	Warrants not exercised
E_T^Q	$\max\{(1 + MK/V_{T_w}^{F*})V_T^{F*} - F^S - F^J, 0\}$	$\max\{V_T^{F*} - F^S - F^J, 0\}$
D_T^J	$\min\{\max\{(1 + MK/V_{T_w}^{F*})V_T^{F*} - F^S, 0\}, F^J\}$	$\min\{\max\{V_T^{F*} - F^S, 0\}, F^J\}$
D_T^S	$\min\{(1 + MK/V_{T_w}^{F*})V_T^{F*}, F^S\}$	$\min\{V_T^{F*}, F^S\}$
W_T	0	0

As in Crouhy and Galai's [7] model, the exercise criteria for warrants is based on firm value. On maturity date T_w immediately after warrant expiration, the stock price is given by

$$S_{T_w} = \begin{cases} \frac{V_{T_w}^{F*} - D_{T_w}^J - D_{T_w}^S + MK}{N + M} \equiv S_{T_w}^W & \text{if warrants exercised} \\ \frac{V_{T_w}^{F*} - D_{T_w}^J - D_{T_w}^S}{N} \equiv S_{T_w}^{NW} & \text{if warrants not exercised} \end{cases},$$

where $D_{T_w}^{J^W}$, $D_{T_w}^{S^W}$ and $D_{T_w}^{J^{NW}}$, $D_{T_w}^{S^{NW}}$ denote values of subordinated and senior debt with and without warrant exercise, respectively. Both $S_{T_w}^W$ and $S_{T_w}^{NW}$ are functions of firm value $V_{T_w}^{F*}$ given that $D_{T_w}^{J^W}$, $D_{T_w}^{S^W}$, $D_{T_w}^{J^{NW}}$, and $D_{T_w}^{S^{NW}}$ are also functions of $V_{T_w}^{F*}$. Let \bar{V}^F be the threshold value of the firm such that $S_{T_w}^W(\bar{V}^F) = K$. If $V_{T_w}^F \geq \bar{V}^F$ the warrant holders should exercise their warrants, otherwise warrant exercise would end up with a negative payoff. The stock price is not the correct signal of exercising warrants for this model. We extend Crouhy and Galai's [7] result by proving the following lemma.

Lemma 1 *Let \bar{V}^F be the value of the firm such that $S_{T_w}^W(\bar{V}^F) = K$. Then $S_{T_w}^{NW}(\bar{V}^F) \geq K$, with equality holding if and only if the firm is entirely equity financed.*

Proof By the definition of the threshold \bar{V}^F , we have

$$\frac{\bar{V}^F - D_{T_w}^{J^W} - D_{T_w}^{S^W} + MK}{N + M} = K$$

$$\Leftrightarrow \bar{V}^F = NK + D_{T_w}^{J^W} + D_{T_w}^{S^W}.$$

Substituting the result above in $S_{T_w}^{NW}$, we have

$$S_{T_w}^{NW}(\bar{V}^F) = \frac{\bar{V}^F - D_{T_w}^{J^{NW}} - D_{T_w}^{S^{NW}}}{N}$$

$$= \frac{(D_{T_w}^{J^W} - D_{T_w}^{J^{NW}}) + (D_{T_w}^{S^W} - D_{T_w}^{S^{NW}})}{N} + K.$$

But we know that $D_{T_w}^{J^W} \geq D_{T_w}^{J^{NW}}$ and $D_{T_w}^{S^W} \geq D_{T_w}^{S^{NW}}$, since the exercise of warrants increases the equity value which reduces the probability of default and hence increases the value of debt. Thus we have shown that $S_{T_w}^{NW}(\bar{V}^F) \geq K$, with equality if and only if the firm is financed entirely by equity (seen by setting the value of all debt equal to zero in the above equation).

By Lemma 1 we see that even if the stock price is higher than the strike K , it might not be optimal for warrants holders to exercise their warrants, i.e., there exists V^F such that $S_{T_w}^{NW}(V^F) > K$ but $S_{T_w}^W(V^F) < K$. The firm value $V_{T_w}^F$ at time T_w would be $V_{T_w}^{F*}$ if warrants were not exercised and $V_{T_w}^{F*} + MK$ if warrants were exercised. Since the value of debt and equity are priced based on the firm value V_t^F , their values right after T_w can be found and are given in Table (2.5).

Table 2.5: Debt and equity values immediately after T_w .

	Warrants exercised ($V_{T_w}^{F*} \geq \bar{V}^F$)	Warrants not exercised ($V_{T_w}^{F*} < \bar{V}^F$)
S_{T_w}	$C(V_{T_w}^{F*} + MK, F^S + F^J, \sigma^F, T - T_w)/(M + N)$	$C(V_{T_w}^{F*}, F^S + F^J, \sigma^F, T - T_w)/N$
$D_{T_w}^J$	$C(V_{T_w}^{F*} + MK, F^S, \sigma^F, T - T_w)$ $-C(V_{T_w}^{F*} + MK, F^S + F^J, \sigma^F, T - T_w)$	$C(V_{T_w}^{F*}, F^S, \sigma^F, T - T_w)$ $-C(V_{T_w}^{F*}, F^S + F^J, \sigma^F, T - T_w)$
$D_{T_w}^S$	$V_{T_w}^{F*} + MK - C(V_{T_w}^{F*} + MK, F^S, \sigma^F, T - T_w)$	$V_{T_w}^{F*} - C(V_{T_w}^{F*}, F^S, \sigma^F, T - T_w)$

Based on Table (2.5), we can compute the value of debt and equity at time $t < T_W$ by the expected discounted value at time T_W under the risk-neutral measure. The time- t stock price is

$$\begin{aligned}
S_t &= e^{-r(T_W-t)} \left\{ E \left[\frac{C(V_{T_W}^{F*} + MK, F^S + F^J, \sigma^F, T - T_W)}{M + N} \right] 1_{(V_{T_W}^{F*} \geq \bar{V}^F)} \right. \\
&\quad \left. + E \left[\frac{C(V_{T_W}^{F*}, F^S + F^J, \sigma^F, T - T_W)}{N} \right] 1_{(V_{T_W}^{F*} < \bar{V}^F)} \right\} \\
&= e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{\bar{Z}}^{+\infty} \frac{C(V_{T_W}^{F*} + MK, F^S + F^J, \sigma^F, T - T_W)}{M + N} \exp \left\{ \frac{-x^2}{2(T_W-t)} \right\} dx \\
&\quad + e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{-\infty}^{\bar{Z}} \frac{C(V_{T_W}^{F*}, F^S + F^J, \sigma^F, T - T_W)}{N} \exp \left\{ \frac{-x^2}{2(T_W-t)} \right\} dx, \quad (2.28)
\end{aligned}$$

where \bar{Z} for this capital structure is defined analogously as in Equation (2.22).

The values of senior and subordinated debt are

$$\begin{aligned}
D_t^S &= e^{-r(T_W-t)} \left\{ E[V_{T_W}^{F*} + MK - C(V_{T_W}^{F*} + MK, F^S, \sigma^F, T - T_W)] 1_{(V_{T_W}^{F*} \geq \bar{V}^F)} \right. \\
&\quad \left. + E[V_{T_W}^{F*} - C(V_{T_W}^{F*}, F^S, \sigma^F, T - T_W)] 1_{(V_{T_W}^{F*} < \bar{V}^F)} \right\} \\
&= e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{\bar{Z}}^{+\infty} (V_{T_W}^{F*} + MK - C(V_{T_W}^{F*} + MK, F^S, \sigma^F, T - T_W)) \\
&\quad \times \exp \left\{ \frac{-x^2}{2(T_W-t)} \right\} dx \\
&\quad + e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{-\infty}^{\bar{Z}} (V_{T_W}^{F*} - C(V_{T_W}^{F*}, F^S, \sigma^F, T - T_W)) \exp \left\{ \frac{-x^2}{2(T_W-t)} \right\} dx, \quad (2.29)
\end{aligned}$$

and

$$\begin{aligned}
D_t^J &= e^{-r(T_W-t)} \left\{ E[C(V_{T_W}^{F*} + MK, F^S, \sigma^F, T - T_W) \right. \\
&\quad \left. - C(V_{T_W}^{F*} + MK, F^S + F^J, \sigma^F, T - T_W)] 1_{(V_{T_W}^{F*} \geq \bar{V}^F)} \right. \\
&\quad \left. + E[C(V_{T_W}^{F*}, F^S, \sigma^F, T - T_W) - C(V_{T_W}^{F*}, F^S + F^J, \sigma^F, T - T_W)] 1_{(V_{T_W}^{F*} < \bar{V}^F)} \right\} \\
&= e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{\bar{Z}}^{+\infty} (C(V_{T_W}^{F*} + MK, F^S, \sigma^F, T - T_W) \\
&\quad - C(V_{T_W}^{F*} + MK, F^S + F^J, \sigma^F, T - T_W)) \exp \left\{ \frac{-x^2}{2(T_W-t)} \right\} dx \\
&\quad + e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \int_{-\infty}^{\bar{Z}} (C(V_{T_W}^{F*}, F^S, \sigma^F, T - T_W) \\
&\quad - C(V_{T_W}^{F*}, F^S + F^J, \sigma^F, T - T_W)) \exp \left\{ \frac{-x^2}{2(T_W-t)} \right\} dx, \quad (2.30)
\end{aligned}$$

respectively.

Since firm value is the sum of the debt, equity and warrant values the warrant price is

$$W_t = (V_t^{F*} - NS_t - D_t^S - D_t^J)/M, \quad (2.31)$$

where S_t , D_t^S , and D_t^J are computed using Equations (2.28), (2.29), and (2.30), respectively.

2.3 Connection Between Firm Value and Share Price

In this section we discuss the relation between share price and debt, warrant and firm values for the capital structures discussed in Section 2.2. Qualitatively, when firm value volatility is constant, it is well known that warrant and debt values both increase with share price — warrants due to basic options properties and debt due to a decrease in the likelihood of default. Thus as the stock price increases, both equity (including warrants) and debt values increase and hence the firm value increases. Conversely, equity, debt and firm values decrease with decreases in share price.

Moreover, it is easily seen that at most one of the stock price volatility or the firm value volatility can be constant. The firm is usually financed by a combination of equity, debt, warrants and other securities depending on its capital structure. Hence the firm value volatility is a combination of the volatility of stock, warrants, debt and other capital structure components. If a firm's stock price is inflated/deflated, the debt-equity ratio changes. Hence the contribution of stock volatility to firm value volatility will also change.

In this section, we provide a method to connect the observable stock price and its volatility to the unobservable firm value and volatility. The pairs (S_t, σ^S) and (V_t^F, σ^F) are connected via a system of two non-linear equations. This connection was first established in Jones, Mason, and Rosenfeld [21] who derived the system for the warrants and common shares capital structure given in Section 2.2.3 and showed that a solution to this system exists. Abinzano and Navas [1] recently extended this connection for the debt, warrants and common shares capital structure given in Section 2.2.4. Here we give the system of equations for the capital structures in Sections 2.2.1, 2.2.2, and 2.2.5. To the best of our knowledge, this is the first appearance of these particular systems of equations in the literature and they are an ancillary contribution of this paper to the financial literature. These systems of equations are the subject of ongoing research.

We start by deriving the coupled system of non-linear equations for a firm with common shares and debt in its capital structure. Derivation of the corresponding systems of equations for other capital structures are based on the same idea and are provided in A.2. Under the model assumptions, firm value follows geometric Brownian motion,

$$dV_t^F = rV_t^F dt + \sigma^F V_t^F dB_t, \quad (2.32)$$

where B_t is a Brownian motion under the risk neutral measure. It can be shown by Ito's formula that

$$\sigma^S = \sigma^F \frac{\partial S_t}{\partial V_t^F} \frac{V_t^F}{S_t}. \quad (2.33)$$

Using Equation (2.6), and $EQ_t = S_t N$, where N is the number of outstanding shares, we can

compute $\frac{\partial S_t}{\partial V_t^F}$, which is

$$\frac{\partial S_t}{\partial V_t^F} = \Phi(d_1)/N, \quad (2.34)$$

where d_1 is given by Equation (2.7). Recall that for this capital structure, the equity value is equal to a European call option written on firm value V_t^F with strike price F , i.e.,

$$NS_t = C(V_t^F, F, \sigma^F, T - t). \quad (2.35)$$

Combining Equations (2.33) and (2.35), we have constructed a system of non-linear equations for V_t^F and σ^F , that is

$$\begin{cases} \sigma^S &= \sigma^F \frac{\Phi(d_1)}{N} \frac{V_t^F}{S_t} \\ NS_t &= C(V_t^F, F, \sigma^F, T - t) \end{cases} \quad (2.36)$$

If S_t and σ^S are given we can determine V_t^F and σ^F using the system of Equations (2.36). This system provides the mechanism for how the effect of fraud observed by a share price change translates to a change in firm value and hence the value of other capital structure securities.

The system of non-linear equations for the capital structure in Section 2.2.2 is very similar to the system of Equations (2.36). We bundle the junior and senior debt as one debt with face value $F^J + F^S$. The system of equations is

$$\begin{cases} \sigma^S &= \sigma^F \frac{\Phi(j_1)}{N} \frac{V_t^F}{S_t} \\ NS_t &= C(V_t^F, F^J + F^S, \sigma^F, T - t), \end{cases} \quad (2.37)$$

where j_1 is given by

$$j_1 = (\log(V_t^F / (F^J + F^S)) + (r + \frac{1}{2}\sigma^{F^2})(T - t)) / (\sigma^F \sqrt{T - t}). \quad (2.38)$$

Similar arguments apply to the capital structures in Section 2.2.4 and Section 2.2.5.

In the rest of this section we illustrate the connection between the pairs (S_t, σ^S) and (V_t^F, σ^F) with some examples. From the systems of Equations in (2.36) and (2.37) it is easily seen that only one of the firm value and stock price volatility can be constant. This fact, along with the effect of firm leverage, gives rise to some seemingly counter-intuitive relationships between the stock price and debt values that are discussed in this section. In Section 2.4 we show the impact of fraud on the value of securities in the capital structure. In order to generate results that agree with intuition (e.g., that a fraud-induced share price drop decreases the value of debt) we fix firm-value volatility as constant and allow the share price volatility to vary. This approach overcomes the model-induced counter-intuitive relationship shown in Section 2.3.1. It can also be argued that firm-value volatility fluctuates less than share price volatility and hence is better approximated by a constant. Unless otherwise stated, the graphs presented in this section and in Section 2.4 use $S_0 = 50$, $\sigma^S = 0.3$, $T - t = 5$, $r = 0.03$ and $N = 100$.

2.3.1 Debt and Equity Capital Structure

In this section, we show the relation between the pairs (V_t^F, σ^F) and (S_t, σ^S) using some numerical results for the debt and equity capital structure. Figure 2.1 shows the relation between

firm value and stock price. We can see from the graph that V_t^F is an increasing function of S_t . Additionally, we see that the effect of share price on firm value is similar (approximately linear with the same slope) across firms with different leverage (face value of debt). Firm value volatility is also an increasing function of stock price assuming that σ^S is held fixed (at 0.3). This relation is illustrated in the right panel of Figure 2.2. We see that share price increases have a smaller effect at higher share prices and that the volatility of highly-levered firms is more sensitive to share price than firms with lower leverage. The left panel of Figure 2.2 uses a constant firm value volatility and shows how the stock volatility changes as a function of the share price. Note that the constant firm value volatility is computed using $\sigma^S = 0.3$ and $S_t = 50$, hence all of the lines on this graph intersect at this point. We see that stock volatility decreases with share price, with the effect decreasing with increased leverage.

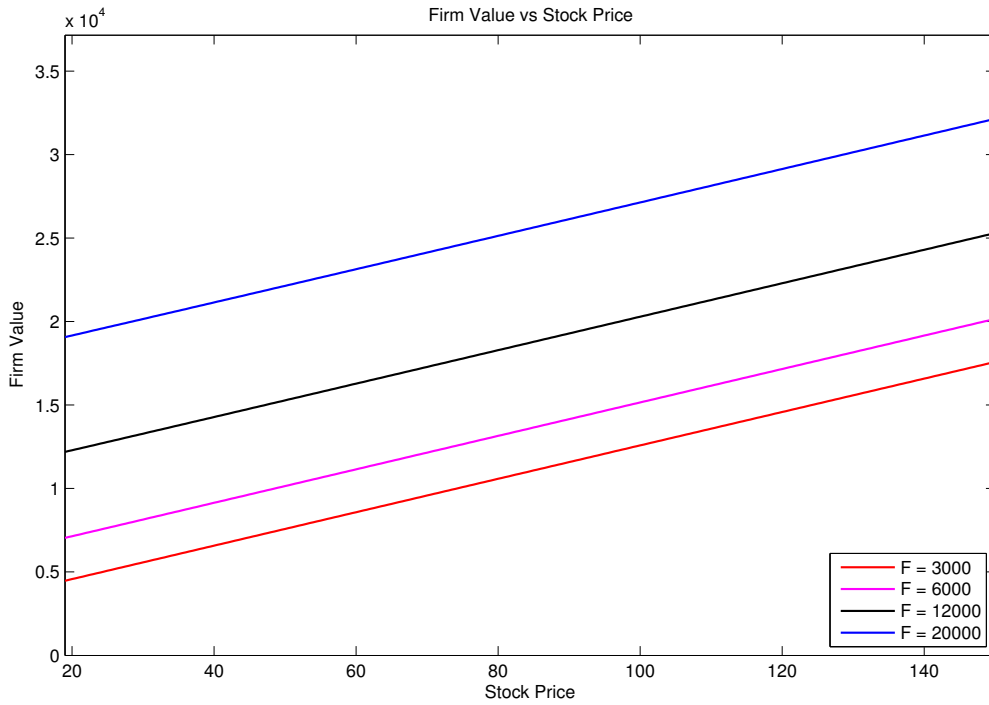


Figure 2.1: Firm value versus stock price for the debt and equity capital structure. F is the face value of the debt and the parameter inputs are $\sigma^S = 0.3$, $T - t = 5$, $r = 0.03$, and $N = 100$.

Figure 2.3 shows the relationship between the debt value and share price, with the different panels of Figure 2.3 corresponding to firms with different leverage as given by debt face values. In the upper left panel we see that debt value increases with share price, with the effect diminishing as the share price increases. This is easily understood as with high share prices the debtholders are quite likely to be paid in full at time T while at low share prices, it is more likely that they will incur a loss. For firms with higher leverage (the other three panels in Figure 2.3), we see that for low share prices that debt value is a decreasing function of share price, then it reaches some minimum value after which the debt value increases with share price.

This is clearly counterintuitive as, other things being equal, the lower the share price, the more likely that losses will be incurred by debtholders at T , hence the lower the value of

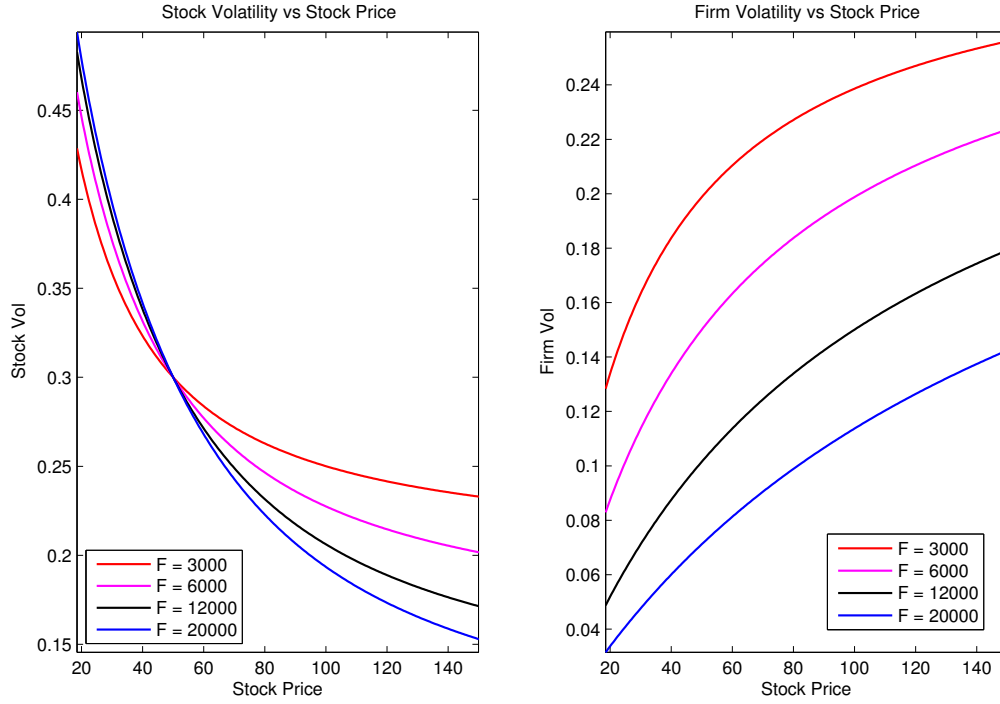


Figure 2.2: Left panel is stock price volatility versus stock price (for fixed firm value volatility) and right panel is firm value volatility versus stock price (for fixed $\sigma^S = 0.3$) for the debt and equity capital structure. F is the face value of the debt and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$.

debt. However, note that for this figure we use a constant stock price volatility of $\sigma^S = 0.3$ and in view of the system of Equations (2.36) we see that the firm-value volatility does not remain constant as the share price changes. In fact, in Figure 2.2 we showed that the firm-value volatility increases with share price. The position of debtholders can be expressed as long a risk-free bond with face value F and short a put option written on firm value, struck at F . As the firm value-volatility decreases (with decrease in stock price), the value of this put option decreases and hence the value of the short put position increases. Thus there are competing effects of share price and firm-value volatility on the value of debt. For low-leverage firms, the share price effect dominates and for more higher-levered firms (which incorporates the combined effect of F and low share prices), the firm-value volatility effect dominates. To confirm this we fix firm-value volatility as constant and let the share price volatility change with the share price (using the same method as used to generate the right panel of Figure 2.2). In all panels of Figure 2.4 we see that the debt value increases with share price for all share price levels, agreeable with intuition. Additionally the value of debt increases with share price until it reaches its discounted face value. For these share prices, default has no chance of occurring hence the debt is risk-free.

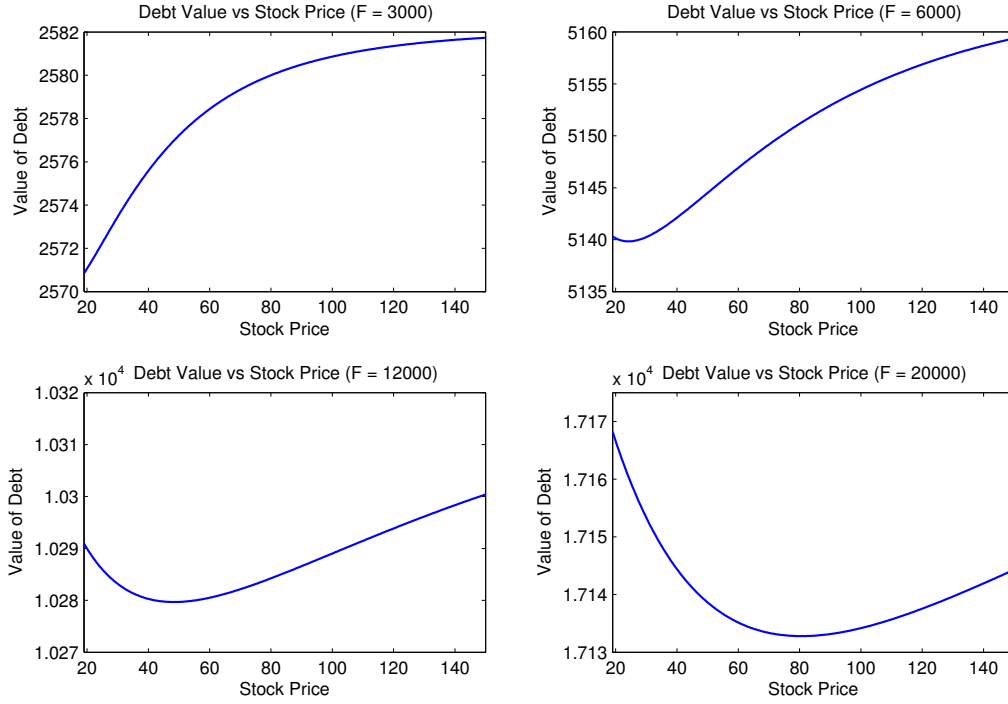


Figure 2.3: Debt value versus stock price for the debt and equity capital structure (fixed stock price volatility). The panels correspond to different debt face values and the parameter inputs are $\sigma^S = 0.3$, $T - t = 5$, $r = 0.03$, and $N = 100$.

2.3.2 Junior and Senior Debt and Equity Capital Structure

For this capital structure, firm value and firm-value volatility are affected in a qualitatively similar way as for the capital structure with just one type of debt. Additionally, the value of senior debt is relatively unaffected by the share price while the value of junior debt behaves similarly to the debt in the previous subsection. In Section 2.4 we see more evidence of this when looking at the effect of fraud-induced share price changes on the value of debt.

2.3.3 Warrants and Common Shares Capital Structure

We show the relation between (V_t^F, σ^F) and (S_t, σ^S) in the pure equity model, in which a firm finances itself by issuing warrants and common shares. We take the warrant expiry time to be 2 years and the strike price is 60. The dilution factor, p , is defined as

$$p = \frac{M}{M + N}, \quad (2.39)$$

where M is the number of warrants issued and N is the number of outstanding shares.

Figure 2.5 shows that V_t^F is an increasing function of S_t in this case. When the dilution factor is higher, the slope of the function is larger, i.e., the firm value increases faster as the stock price increases. This is because the number of outstanding shares is fixed and increasing

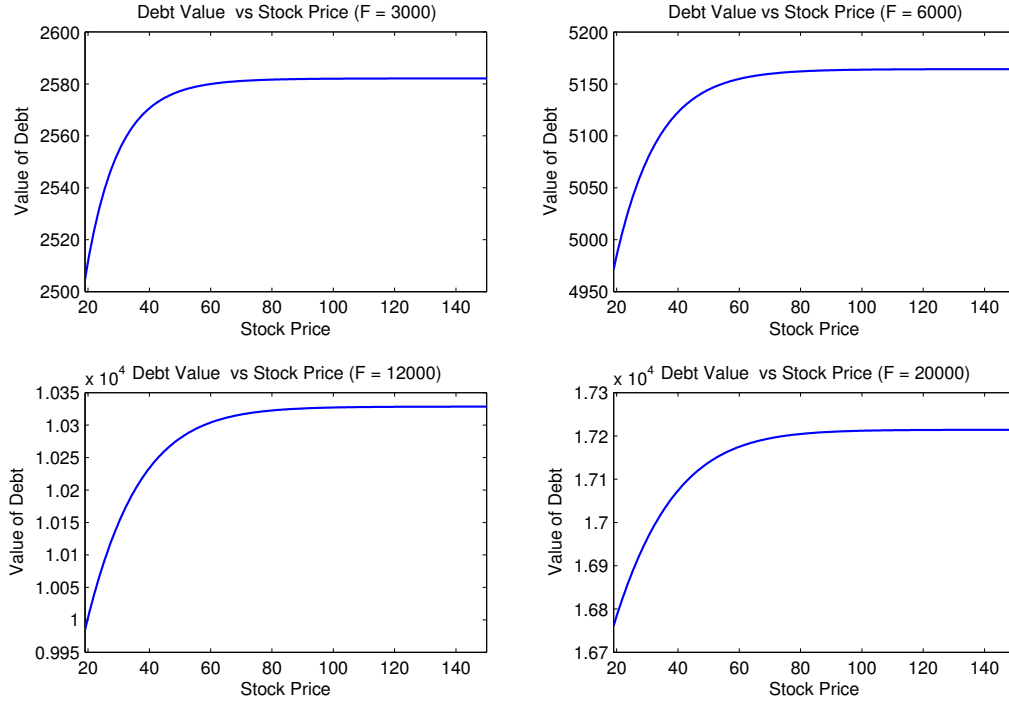


Figure 2.4: Debt value versus stock price for the debt and equity capital structure (fixed firm value volatility). The panels correspond to different debt face values and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$. Firm-value volatility is computed using $S_t = 50$ and $\sigma^S = 0.3$.

p amounts to increasing the number of warrants. Figure 2.5 was generated with a fixed stock volatility and varying firm-value volatility. Figure 2.6 gives the warrant price as a function of share price (constant firm-value volatility). For this capital structure we see that the dilution factor has little effect on warrant value, with the main determinant being stock price.

The firm value volatility is a hump shaped function of S_t , as shown in the right panel of Figure 2.7. The volatility of firm value reaches its maximum when the stock price is a little above the strike price of 60. The firm value volatility is much more sensitive to share price changes when the dilution factor is high. The left panel of Figure 2.7 plots the stock volatility as a function of share price for a fixed firm value volatility (computed at $\sigma^S = 0.3$ and $S_t = 50$). We see that σ^S decreases for lower share prices, achieves a minimum, then increases. Additionally the changes in stock volatility are greater with increasing dilution factor.

2.3.4 Debt, Warrants and Common Shares Capital Structure

Here we illustrate the relationship between firm, warrant, and debt values, stock volatility and stock price for a firm that is financed with a single type of debt, warrants and common shares. For all figures presented in this subsection, the firm value volatility is held constant and, as before, is computed for $\sigma^S = 0.3$ and $S_t = 50$. The left and right panels of Figure 2.8 plot firm value as a function of stock price for dilution factors of $p = 1/6$ and $p = 1/2$, respectively.

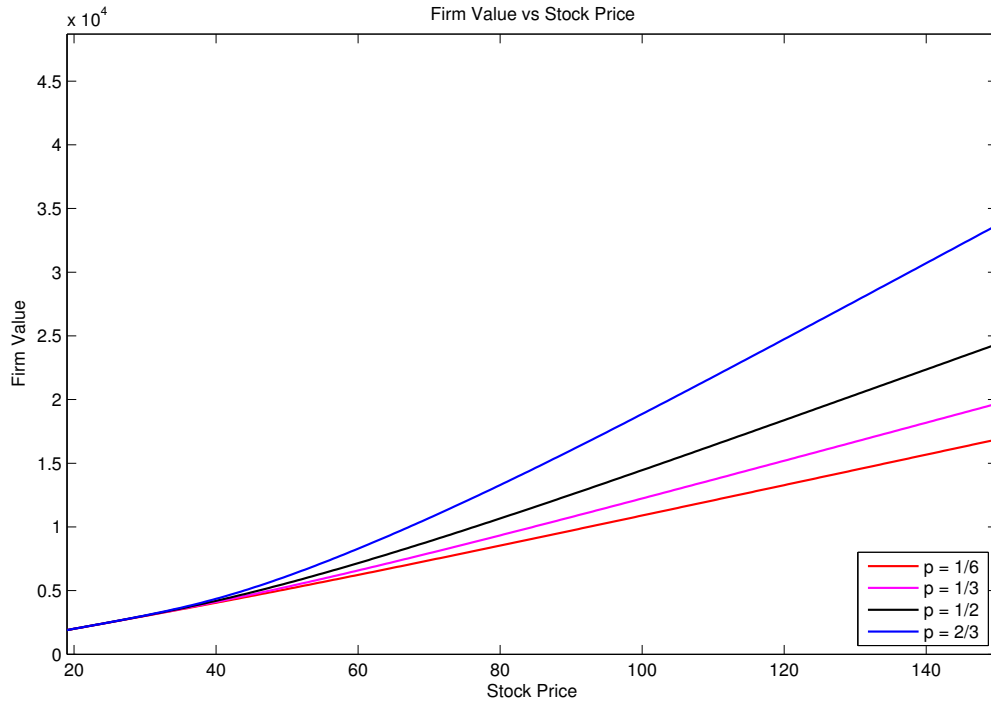


Figure 2.5: Firm value versus stock price for the warrants and common shares capital structure. p is the dilution factor and the parameter inputs are $\sigma^S = 0.3$, $T_W - t = 2$, $r = 0.03$, $K = 60$ and $N = 100$.

Firm value increases with share price and, due to a higher number of warrants, it increases faster for a higher dilution factor. This observation is consistent across debt face values.

Figure 2.9 plots debt value versus stock price, with the four panels corresponding to different firm leverages, as indicated by the debt face value. We see that the debt value increases with stock price, reflecting the diminished probability of default at higher stock prices (compare with Figure 2.4, the analogous graphs for a debt and equity financed firm). There is clearly a dilution effect, with the debt from the high-dilution firm ($p = 1/2$) being less valuable and more sensitive to stock price changes than for the low-dilution firm ($p = 1/6$). A reason for this dilution effect is that the constant firm value volatility computed using $\sigma^S = 0.3$, $S_t = 50$ is higher for the high-dilution firm ($p = 1/2$) than the low-dilution firm ($p = 1/6$).

Note that debt can be viewed as a long position in a risk-free bond with face value F and short a put option written on firm value and struck at F . The value of a put option increases with volatility, hence the short put option with a higher volatility gives a lower debt value. Additionally, the warrant exercise criteria depends on the firm value not just the stock price and the critical values for these differ between the low and high dilution firms. The stock price level that triggers warrant exercise is higher for the high-dilution firm than the low-dilution firm. Upon exercise of the warrants, there is a cash infusion that increases the value of the firm, including the debt. Warrants from the high-dilution firm are less likely to be exercised, resulting in lower firm values and debt values than the low-dilution firm.

The left and right panels of Figure 2.10 plot warrant price versus stock price for dilution

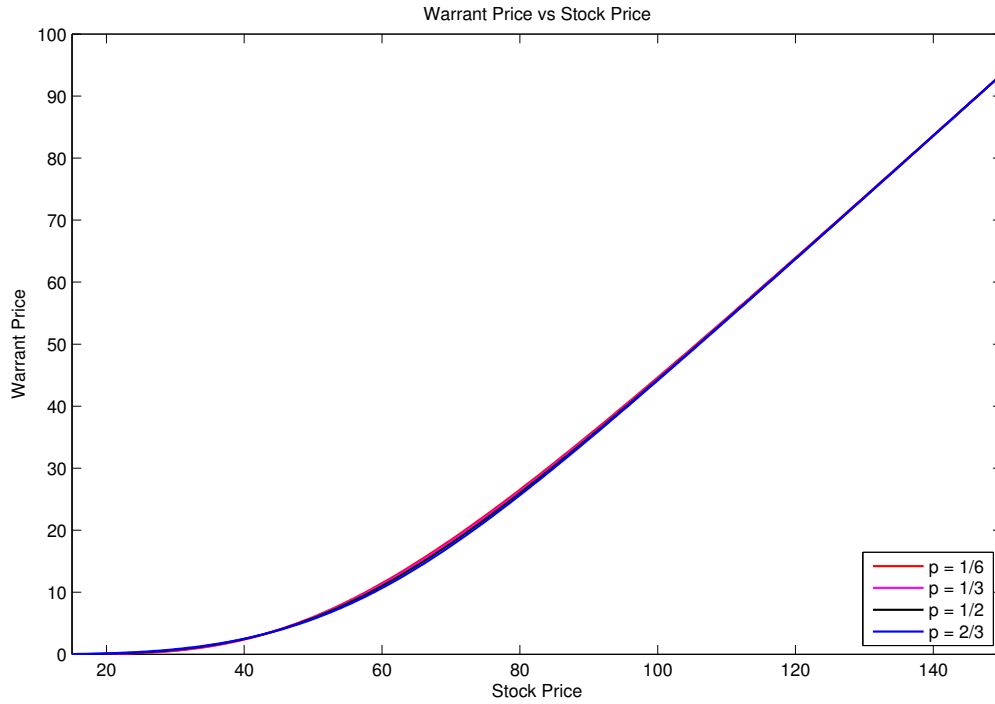


Figure 2.6: Warrant value versus stock price for the warrants and common shares capital structure (constant firm value volatility). p is the dilution factor and the parameter inputs are $T_W - t = 2$, $r = 0.03$, $K = 60$ and $N = 100$. Firm-value volatility is computed using $S_t = 50$ and $\sigma^S = 0.3$.

factors $p = 1/6$ and $p = 1/2$, respectively. Warrant price increases at an increasing rate with stock price and both the dilution factor and leverage (debt face value) have little effect on warrant value.

With firm value volatility fixed (computed using $\sigma^S = 0.3$, $S_t = 50$) Figure 2.11 plots the stock volatility versus stock price, with the left and right panels corresponding to $p = 1/6$ and $p = 1/2$, respectively. For $p = 1/6$ we find σ^S is a decreasing function of stock price, with a decreasing rate. For $p = 1/2$, σ^S is almost always a decreasing function of stock price. However, for firms with low leverage ($F = 3000$ and $F = 6000$), the stock price volatility increases at high share prices. For both dilution factor values there is a leverage effect.

2.3.5 Junior and Senior Debt, Warrants and Common Shares Capital Structure

For this capital structure, firm value and warrant values are affected by the stock price in a qualitatively similar way as for the capital structure with warrants, common shares and one type of debt. Additionally, the value of senior debt is relatively unaffected by the share price while the value of junior debt behaves similarly to the debt in the previous subsection. In Section 2.4 we see more evidence of this when looking at the effect of fraud-induced share price changes on the value of junior and senior debt and warrants.

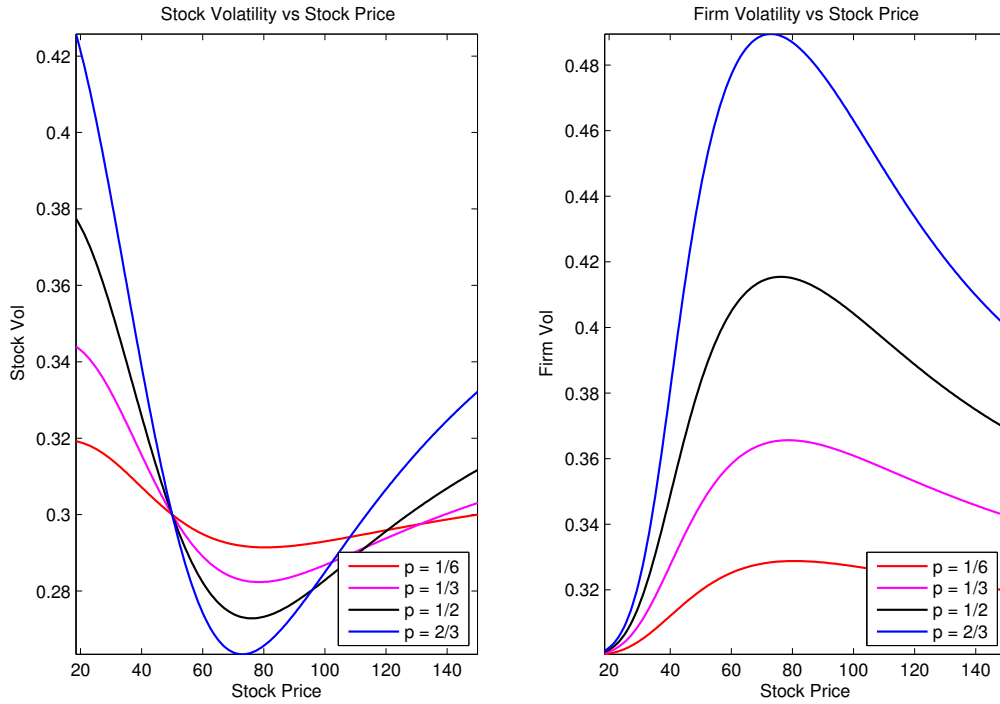


Figure 2.7: Left panel is stock price volatility versus stock price (fixed firm value volatility) and right panel is firm-value volatility versus stock price (fixed stock price volatility) for the warrants and common shares capital structure. p is the dilution factor and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$. Firm-value volatility for the right panel is computed using $S_t = 50$ and $\sigma^S = 0.3$.

2.4 Effect of Fraud on Securities Value

In this section we show how fraud affects the value of securities issued by the firm for the capital structures previously discussed. Note that here we fix the firm value volatility and allow the stock price volatility to vary. Hence the counter-intuitive effect of share price on debt values shown in Figure 2.3, for example, are avoided in these results. Unless otherwise noted, the base parameter values used in the previous section are also used here. Additionally for each of the capital structures, we look at the effect of fraud-induced share price changes for $S_t = 50$ and $S_t = 125$. Note that

$$\delta = \frac{\tilde{S}_t}{S_t} \quad (2.40)$$

and we plot the ratio of the “true” (indicated with a tilde) to “fraudulent” values of the securities as a function of δ . Values of δ greater than 1 indicate that the fraud depressed the share price (not common) and values of δ less than one correspond to fraud inflating the share price (very common). If $\delta = 1$ then there is no misrepresentation. Note that the duration of class period can impact the size of settlements. The longer the class period typically means that more non-round-trip transactions happened during the class period, but it does not necessarily imply a wider damages ribbon.

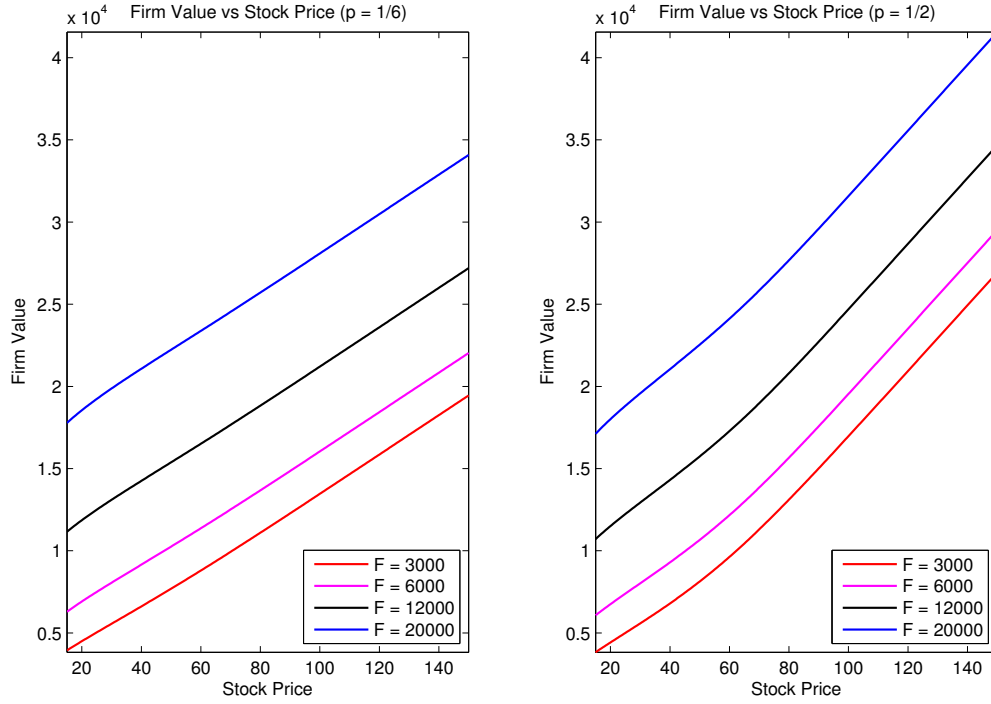


Figure 2.8: Left and right panels are firm value versus stock price (fixed firm value volatility) for the debt, warrant and common shares capital structure for $p = 1/6$ and $p = 1/2$, respectively. F is the face value of the debt and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$. Firm-value volatility is computed using $S_t = 50$ and $\sigma^S = 0.3$.

2.4.1 Debt and Equity Capital Structure

Figure 2.12 plots the ratio $\frac{\tilde{D}}{D}$ as a function of δ for $S_t = 50$ (left panel) and $S_t = 125$ (right panel), respectively. We note that the lines for all values of F (face value of debt) intersect at $\delta = 1$, the case of no fraud. The ratio $\frac{\tilde{D}}{D}$ increases at a decreasing rate with δ . This makes sense as for δ much bigger than 1, it is unlikely that debtholders will incur losses and for δ smaller than one, the reverse holds. There is an obvious leverage effect, with firms having less leverage (low F or high S) having debt values less affected by fraud than firms with more leverage (high F or low S).

2.4.2 Junior and Senior Debt and Equity Capital Structure

Figure 2.13 plots the true/fraudulent value ratios of junior (left panels) and senior (right panels) debt as a function of δ for $S_t = 50$ (top panels) and $S_t = 125$ (bottom panels), respectively. We see that the junior debt value behaves much like the debt for a firm financed with only one type of debt and equity (previous subsection). The leverage effect discussed in Section 2.4.1 is also evident in the relative values of junior debt here. Additionally, the value of senior debt is unaffected by the fraud-induced change in stock price and this is true across firm leverage. This makes sense as the senior debt will only incur losses if the junior debt is completely wiped

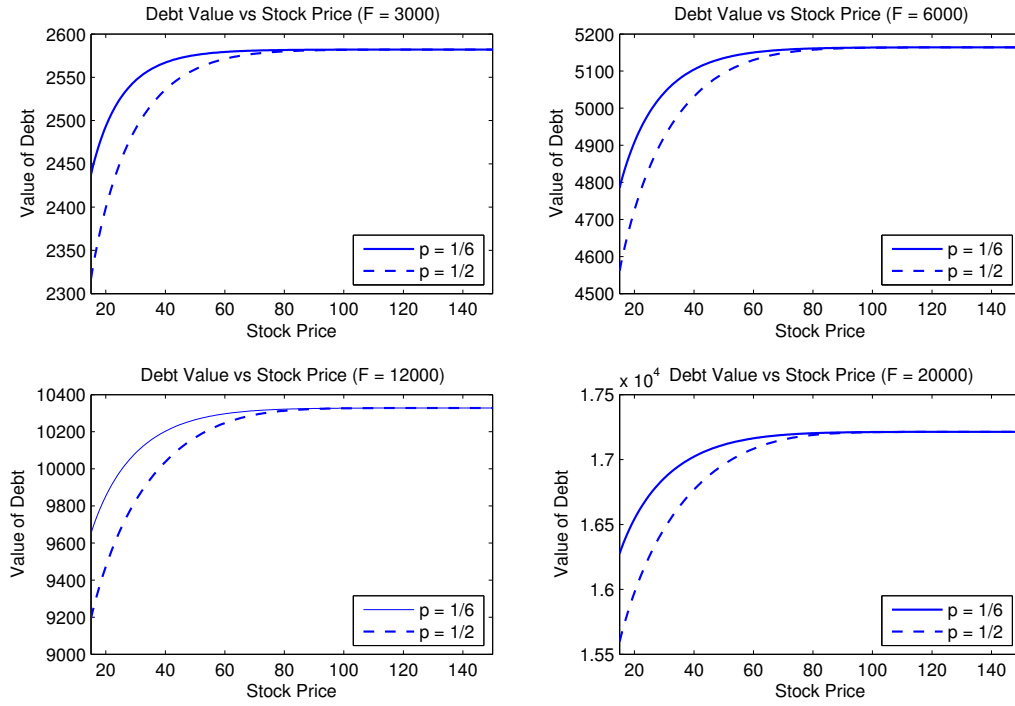


Figure 2.9: Debt value versus stock price for the debt, warrant and common shares capital structure (fixed firm-value volatility). The panels correspond to different debt face values and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$. Firm-value volatility is computed using $S_t = 50$ and $\sigma^S = 0.3$.

out, a highly unlikely circumstance given the parameter values and structures used. For small values of δ we see evidence that the value of senior debt for face values 1500 and 3000 is affected by fraud. This is due to the fact that the junior debt face values are also small (1500 and 3000), hence providing less of a loss-absorbing cushion than for higher face values.

2.4.3 Warrants and Common Shares Capital Structure

Figure 2.14 shows the relationship between $\frac{\bar{W}}{W}$ and δ for $S_t = 50$ (left panel) and $S_t = 125$ (right panel). The warrant maturity is 2 years and the strike price is 60. The right panel shows that for deep in the money warrants, there is no dilution effect on the warrant value as δ changes. The left panel shows that for warrants close to the money, there is a mild dilution effect for very high δ . Additionally, for deep-in-the-money warrants, the relative value changes much less with δ than for at-the-money warrants. This is due to the fact that it is certain the warrants will be exercised, no matter how large/small the relative change in share price.

2.4.4 Debt, Warrants and Common Shares Capital Structure

Figure 2.15 plots $\frac{\bar{D}}{D}$ as a function of δ for $S_t = 50$ (left panels), $S_t = 125$ (right panels), $p = 1/6$ (top panels) and $p = 1/2$ (bottom panels), respectively. As with the debt and eq-

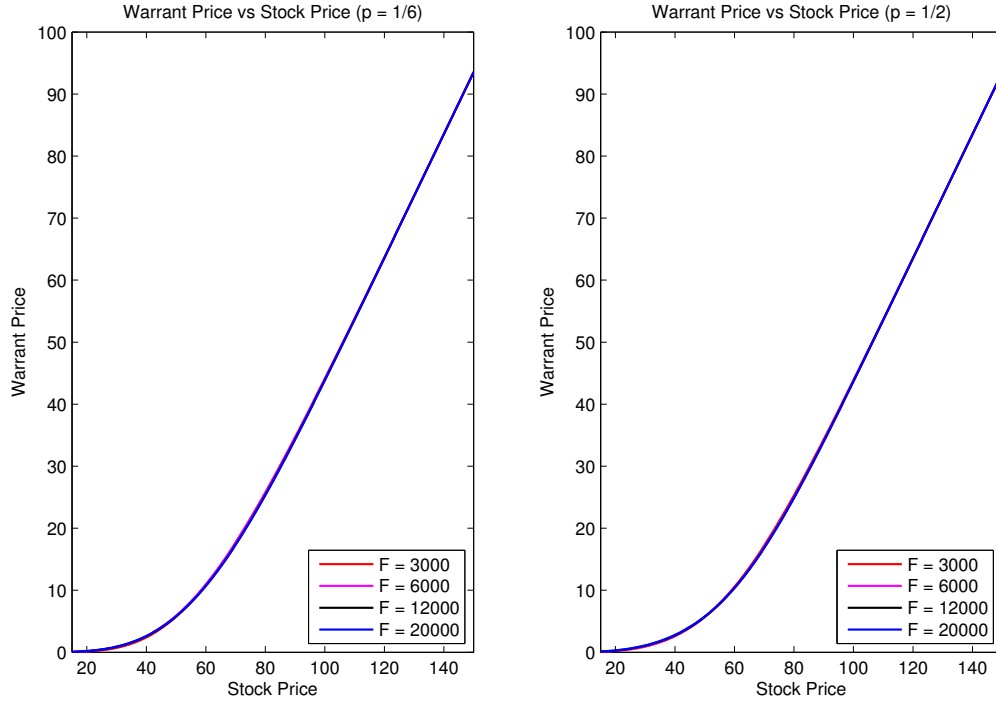


Figure 2.10: Left and right panels are warrant value versus stock price (fixed firm-value volatility) for the debt, warrant and common shares capital structure for $p = 1/6$ and $p = 1/2$, respectively. F is the face value of the debt and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$. Firm-value volatility is computed using $S_t = 50$ and $\sigma^S = 0.3$.

uity capital structure (Section 2.4.1) the ratio \tilde{D}/D increases at a decreasing rate with δ and a similar leverage effect is observed — lower-leverage firms are less affected by fraud than higher-leverage firms. Interestingly, the dilution factor does not seem to have much effect on the ratio \tilde{D}/D .

Figure 2.16 plots the ratio $\frac{\tilde{W}}{W}$ as a function of δ for $S_t = 50$ (left panels), $S_t = 125$ (right panels), $p = 1/6$ (top panels) and $p = 1/2$ (bottom panels), respectively. Here the impact of fraud on warrant value is the same across warrant moneyness, leverage (S_t and F), and dilution ratios. This finding for the dilution ratio agrees with the findings for the firm financed only by warrants and common shares. However, the finding for the effect of S_t differs due to the presence of debt in the capital structure.

2.4.5 Junior and Senior Debt, Warrants and Common Shares Capital Structure

Figure 2.17 plots the true/fraudulent value junior debt ratios as a function of δ for $S_t = 50$ (left panels), $S_t = 125$ (right panels), $p = 1/6$ (top panels) and $p = 1/2$ (bottom panels), respectively. Unsurprisingly, The ratio behaves similarly to the debt value ratio in the previous subsection. There appears to be a similar leverage effect and the dilution factor is unimportant.

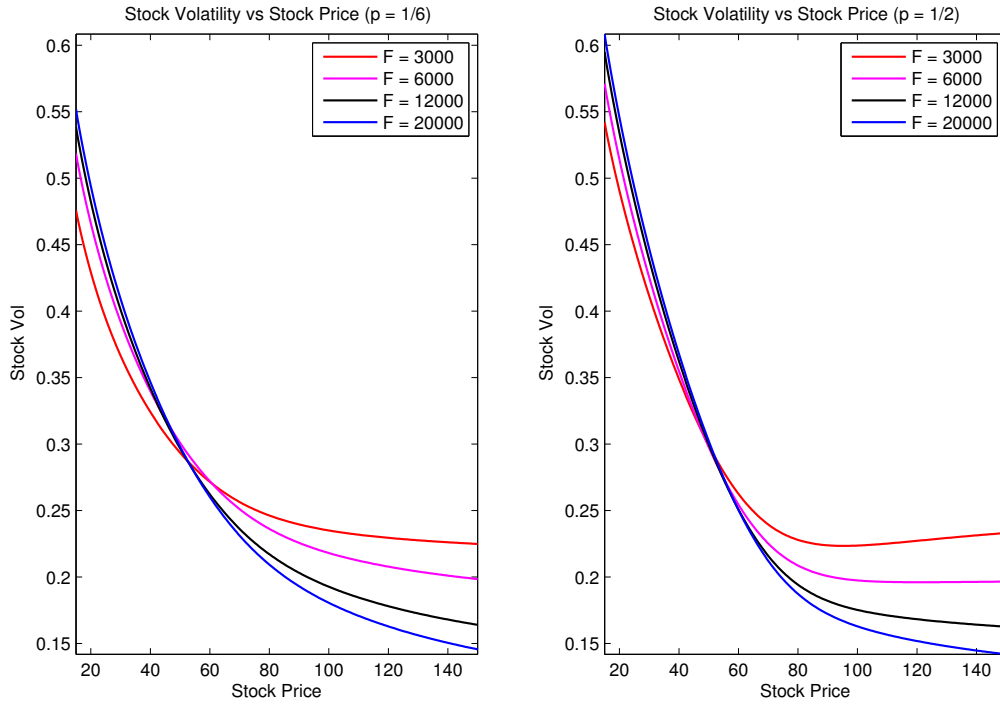


Figure 2.11: Left and right panels are stock price volatility versus stock price (for fixed firm value volatility) for the debt, equity, and common shares capital structure for $p = 1/6$ and $p = 1/2$, respectively. F is the face value of the debt and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$. Firm-value volatility is computed using $S_t = 50$ and $\sigma^S = 0.3$.

Figure 2.18 is a similar plot, but for the relative value of senior debt. As with the other capital structure with both types of debt studied here, the senior debt value is much less affected by the fraud than the junior debt values, due to the loss-absorbing cushion provided by the junior debt. However for this capital structure and for very small values of δ (i.e., large drop in share price after the misrepresentation is revealed), there is a small drop in senior debt values. Note that for this capital structure with two types of debt, the ratio $\frac{\bar{W}}{W}$ behaves similarly as in the previous subsection (capital structure with one type of debt).

2.5 Agnico-Eagle Mines Ltd. Case Study

In this section, we demonstrate our damages computation methodology with a case study. Agnico-Eagle Mines Ltd. (AEM) is a mining company (listed in Toronto Stock Exchange (TSE) and New York Stock Exchange (NYSE)) with operations in Canada, Finland, Mexico and the United States. In March 2012, secondary market securities class actions were filed in Ontario and Quebec against AEM and certain of AEM's current and former officers and directors. The actions allege that AEM failed to disclose a water inflow issue at its Goldex mine (in Quebec) during the class period — March 26, 2010 to October 18, 2011 (indicated by the vertical red lines in Figure 2.19). In Figure 2.19, the stock price drops significantly

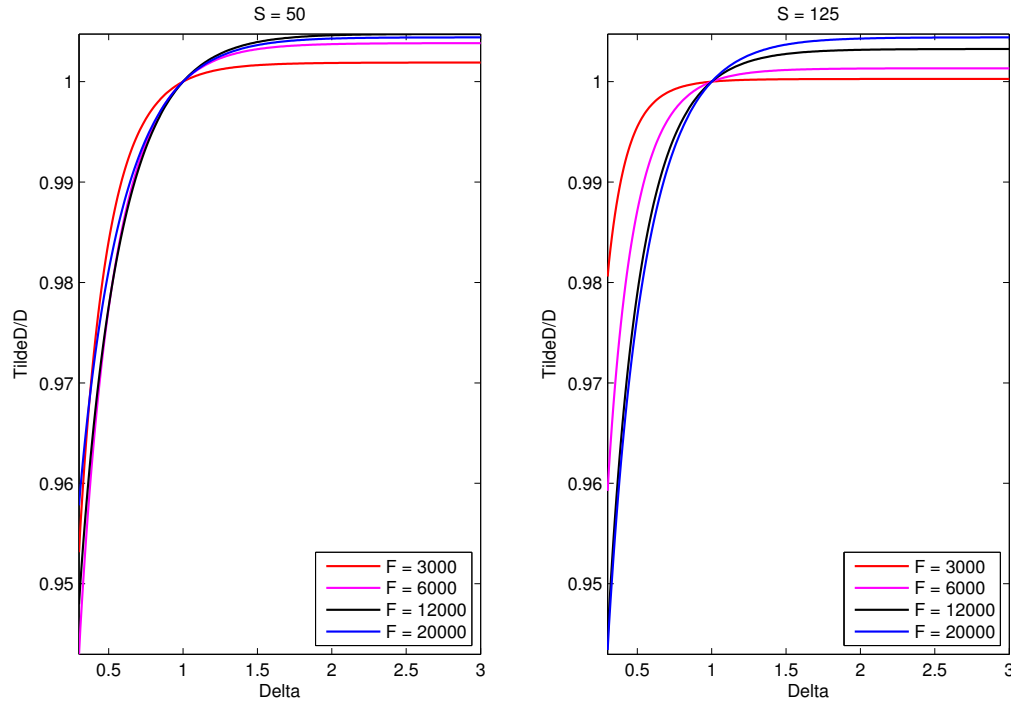


Figure 2.12: Left and right panels plot $\frac{\tilde{D}}{D}$ versus δ for the debt and equity capital structure for $S_t = 50$ and $S_t = 125$, respectively. Lines in each plot correspond to different debt face values, F , and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$.

on the fraud disclosure date (October 18, 2011) indicating that the share price was inflated during the class period. The common share value line was calculated using the event study approach as described in Section 2.1.1. The S&P/TSX Composite Total Return Index and the S&P/TSX Gold Total Return Index are used as the market return R_{Mt} and the industry return R_{It} , respectively. Two years of daily observations from January 04, 2010 to December 30, 2011 are used to estimate the parameters in the econometric model. On October 18, 2011, there was a price change of \$10.59 of which \$7.03 was attributed to the revelation of fraud after controlling for market and industry factors.

The capital structure of AEM during the class period is detailed as follow. The financial statements of AEM indicate that long-term liabilities include two types of long-term debts — bank credit facilities and notes (see [2]). A credit facility is a loan in the form of revolving credit, in which the customer is allowed to borrow/repay funds as needed. Short-term liabilities and the other long-term liabilities (e.g., deferred income and mining tax liabilities) are ignored in our analysis. AEM equity was comprised of 156.7 million common shares, 8.4 million employee stock options, and 8.6 million warrants on March 31, 2010, the start of the class period.⁵ In this case study, we make the simplification that AEM equity is comprised of common shares, warrants and two types of long-term debt. On June 22, 2010 (during the class

⁵During the class period, the numbers of common shares and employee stock options changed, while the number of warrants was unchanged. The number of common shares increased from 156.7 million to 169.2 million.

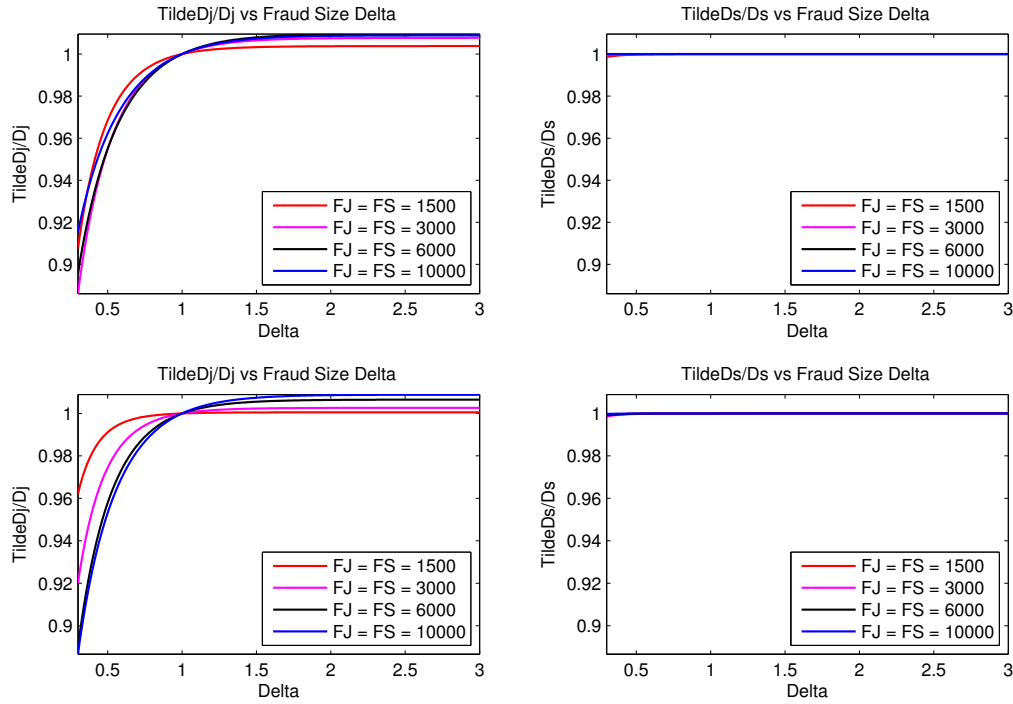


Figure 2.13: True/fraudulent value ratios of junior (left panels) and senior (right panels) debt as a function of δ for $S_t = 50$ (top panels) and $S_t = 125$ (bottom panels), respectively, for the junior and senior debt and equity capital structure. Lines in each plot correspond to different debt face values, F , and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$.

period), AEM amended its credit facilities: the amount available increased to \$1.2 billion and the maturity date extended to June 22, 2014. Details of the interest rate paid on the credit facilities are not available. On April 7, 2010, the company closed a note offering with institutional investors in the U.S. and Canada for a private placement of \$600 million of guaranteed senior unsecured notes due in 2017, 2020 and 2022. The notes had a weighted average maturity of 9.84 years and weighted average yield of 6.59% at issuance. Proceeds from the notes were used to repay amounts owed under the company's then outstanding credit facilities. The credit facilities and the notes rank equally in seniority. Details of the coupon payments and face value of the notes proved difficult to obtain. In this case study, we treat the notes and credit facilities as one zero coupon debt with the weighted average maturity and calculate its damages ribbon during the class period.

A step-by-step implementation of the damages calculation is given below. Based on the capital structure of AEM during the class period, we decide to use the capital structure model discussed in Section 2.3.4, where firm value consist of common shares, warrants and a zero coupon debt. We first transform AEM's two types of long-term debts into two coupon bonds. The first bond corresponds to the credit facilities and is assigned a face value of \$ 69.29 million — the weighted average amount drawn from the credit facilities during the class period. The maturity date of this bond is June 22, 2014. The second bond corresponds to the notes and is given a face value of \$600 million — the total face value of the issued notes. The maturity date

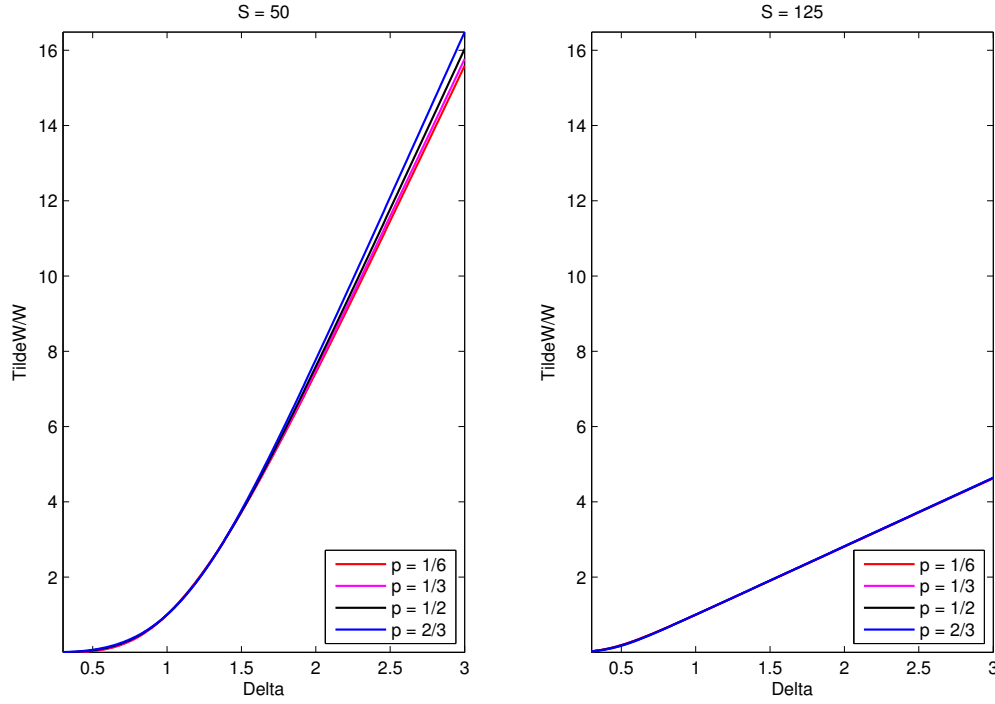


Figure 2.14: Left and right panels plot $\frac{\tilde{W}}{W}$ versus δ for the warrant and common shares capital structure for $S_t = 50$ and $S_t = 125$, respectively. Lines in each plot correspond to different dilution factors, p , and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$.

of this bond is Feb 8, 2020, the weighted average maturity date of the three notes. Since details of the interest and coupon payments are not available, both bonds are assumed to pay annual coupons of 6.59% (the average yield of the notes), with coupon payments made at the end of each year. In order to use the structural model under Merton framework, we transform the two coupon bonds into one zero coupon debt; weighted average maturity of debt and its face value are calculated for each date within the class period. Given the share price value line, we construct the value line of warrant and debt (equivalently the damages ribbon for warrant and debt). The true value of warrant and debt on a date i during the class period can be calculated using the following steps:

- 1 Calculate the true value of equity \tilde{S}_i using the share price value line as described in Section 2.1.1;
- 2 Calculate the firm value V_i^F and firm value volatility σ_i^F from observed share price S_i and stock volatility σ_i^S by solving non-linear system (A.8);
- 3 Compute the true firm value $\tilde{V}_i^F = v_{\tilde{S}_i}(\sigma_i^F)$ by solving the second equation of the non-linear system⁶ (A.8);
- 4 Calculate the damages ribbon for warrant and debt:

⁶The second equation in the non-linear system depends only on firm volatility but not on stock volatility.

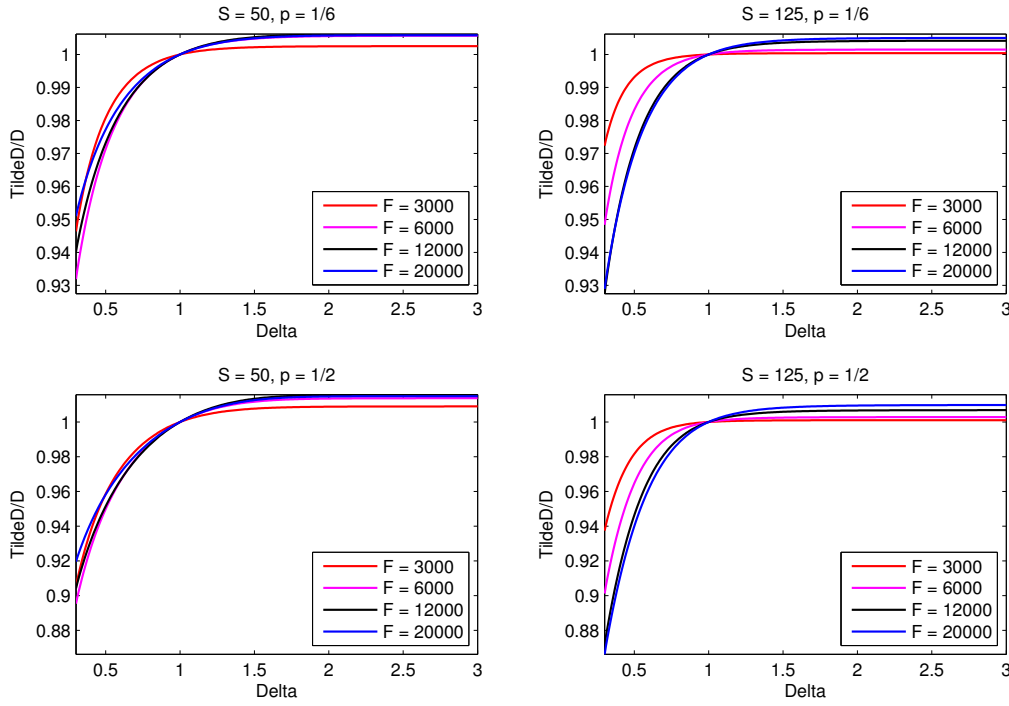


Figure 2.15: $\frac{\tilde{D}}{D}$ versus δ for $S_t = 50$ (left panels), $S_t = 125$ (right panels), $p = 1/6$ (top panels) and $p = 1/2$ (bottom panels), respectively, for the debt, warrants and common shares capital structure. Lines in each plot correspond to different debt face values, F , and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$.

- a Calculate the debt values $D(V_i^F, \sigma_i^F)$ and $D(\tilde{V}_i^F, \sigma_i^F)$ based on the firm values V_i^F and \tilde{V}_i^F , respectively, using equation (2.24). And calculate the date- i damages ribbon of debt $R_i^D = D(V_i^F, \sigma_i^F) - D(\tilde{V}_i^F, \sigma_i^F)$.
- b Calculate the warrant values $W(V_i^F, \sigma_i^F)$ and $W(\tilde{V}_i^F, \sigma_i^F)$ based on the firm values V_i^F and \tilde{V}_i^F , respectively, using equation (2.25). And calculate the date- i damages ribbon of warrant $R_i^W = W(V_i^F, \sigma_i^F) - W(\tilde{V}_i^F, \sigma_i^F)$.

5 The true value of warrant \tilde{W}_i is the difference between the observed warrant price W_i and R_i^W , i.e., $\tilde{W}_i = W_i - R_i^W$.

Figure 2.20 shows the warrant value line constructed using the steps described above. Figure 2.21 shows the damage ribbon of debt calculated using our proposed methodology. Regarding the other model parameters, the risk-free rate is taken as 3.72%, the average U.S. Treasury long-term composite rate (The unweighted average of bid yields on all outstanding fixed-coupon bonds neither due nor callable in less than 10 years) from April 7, 2010 to October 18, 2011. The stock volatility is the 90 days historical volatility obtained from Bloomberg⁷. Data of the number of outstanding shares and the number of outstanding warrants is downloaded from Bloomberg. The value line/damages ribbon shown in Figure 2.20 and 2.21 for warrant and debt,

⁷It is the volatility of the log return of stock price; it's estimated using 90 days share price

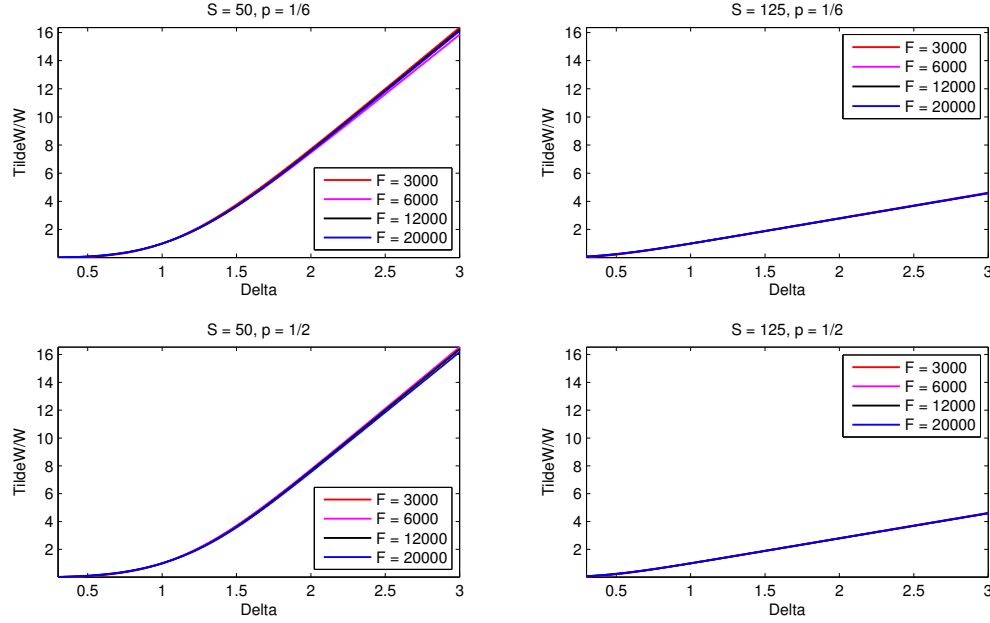


Figure 2.16: $\frac{\tilde{W}}{W}$ versus δ for $S_t = 50$ (left panels), $S_t = 125$ (right panels), $p = 1/6$ (top panels) and $p = 1/2$ (bottom panels), respectively, for the debt, warrants and common shares capital structure. Lines in each plot correspond to different debt face values, F , and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$.

respectively, can be used to assess damages in a security class action. Figure 2.21 shows that the damage ribbons calculated using the Merton model is thinner than that using the First Passage Time (FPT) model [31]. This is in line with the results of empirical study [10], which shows that the Merton model fails to generate high enough corporate bond spreads as compared to those observed in the markets, while the FPT model overstates the observed spreads in general. Given that the debt value under the FPT model is more sensitive to leverage change than the Merton model, the damages ribbon of debt (which is the difference in debt value between the observed leverage and the “true” leverage) under the Merton model is thinner than that under the FPT model.

2.6 Conclusion

Using a modified Merton framework for valuing corporate securities and a connection between the observable share price and firm value we show the impact of a fraud-induced share price change on the value of the other corporate securities in the capital structure. We show that the impact on debt value depends on firm leverage and debt seniority and not on the warrant dilution factor. Generally, the debt for higher-leverage firms is more sensitive to the misrepresentation impact than lower-leverage firms and junior debt is more affected by fraud than senior debt. The impact on warrant value is determined by warrant moneyness (stock price), with the dilution factor having no effect. Moreover, we demonstrate by a case study the implementation

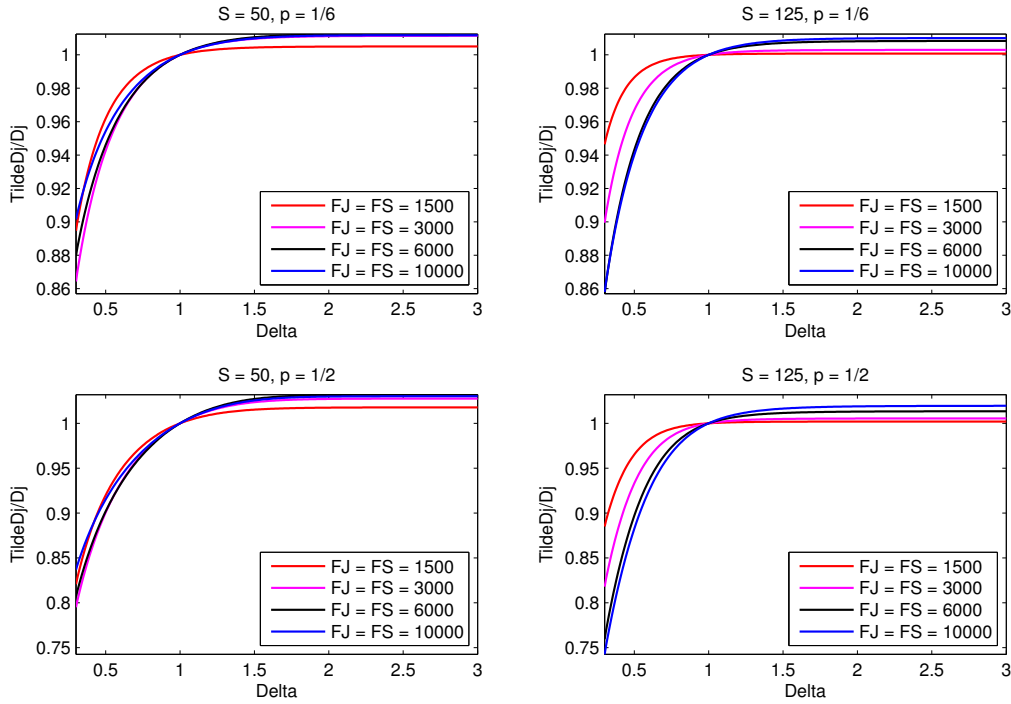


Figure 2.17: True/fraudulent value junior debt ratios as a function of δ for $S_t = 50$ (left panels), $S_t = 125$ (right panels), $p = 1/6$ (top panels) and $p = 1/2$ (bottom panels), respectively, for the junior and senior debt, warrants and common shares capital structure. Lines in each plot correspond to different debt face values, F , and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$.

of a damages calculation methodology, which is consistent with the event study approach for equities damages calculation, for warrants and debts.

This study is the first step in using a structural modelling approach to compute damages due to misrepresentation to holders of warrants and debt. It can be relevant not only for estimating potential damages, but in the fair allocation of damage awards across holders of the fraudulent firm's securities. Additionally, many trading strategies involve positions in more than one security issued by a single firm. For example, to hedge a credit-risky long bond position, one can short the shares of the issuing firm. For such a position, damages due to fraud must be computed on a net basis and hence to do this accurately, one must compute the change in both stock and bond values due to fraud, something this modelling framework allows.

In addition to our main findings, we provide ancillary contributions to the warrant valuation and capital structure literature. In Section 2.2.5 we extend Crouhy and Galai's [7] framework to a capital structure that includes both junior and senior debt and in Lemma 1 discuss the correct warrant exercise criteria for this structure. Additionally, we broaden the connection between observable stock price and its volatility with unobservable firm value and its volatility discussed in Ukhov[30] and Abinzano and Navas[1] to other capital structures not previously considered.

There are many different avenues for future research. Extending the modelling framework

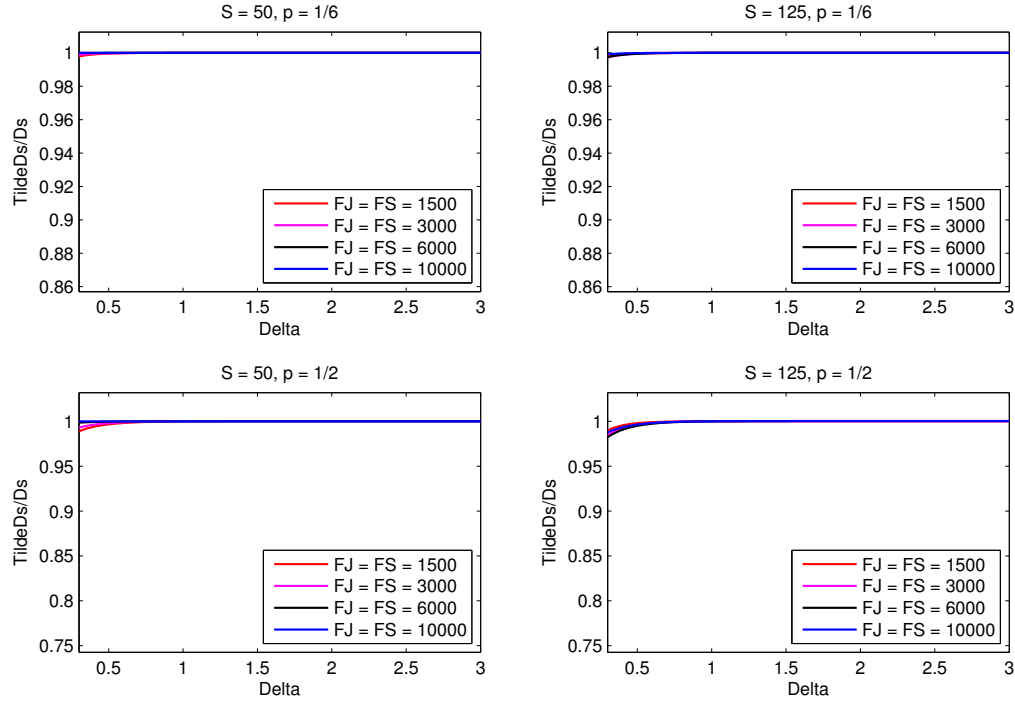


Figure 2.18: True/fraudulent value senior debt ratios as a function of δ for $S_t = 50$ (left panels), $S_t = 125$ (right panels), $p = 1/6$ (top panels) and $p = 1/2$ (bottom panels), respectively, for the junior and senior debt, warrants and common shares capital structure. Lines in each plot correspond to different debt face values, F , and the parameter inputs are $T - t = 5$, $r = 0.03$, and $N = 100$.

to more realistic capital structures (e.g., allowing for coupon bonds) and to the infinite and finite time horizon first-passage-time paradigms (for bond defaults) are obvious directions to pursue. Using this extended modelling framework, an empirical analysis of the model performance would be required to see how well the model-predicted security value changes match the observed changes.

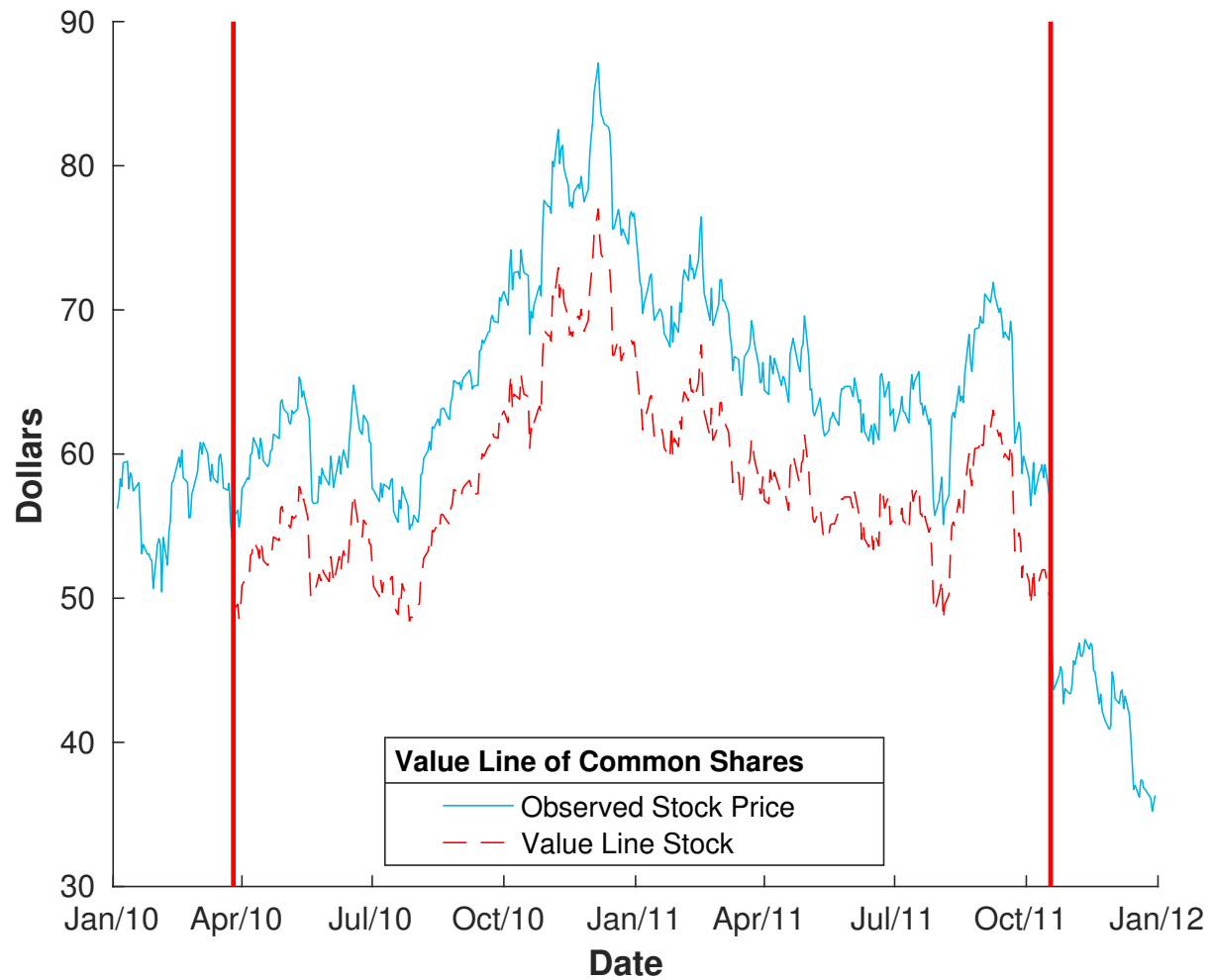


Figure 2.19: Value line of common shares - The period between the two vertical lines in is the class period. The blue solid line shows the historical stock price of AEM from January 2010 to December 2011. The stock price value line, represented by the red dashed line, is constructed by the event study approach discussed in Section 2.1.1

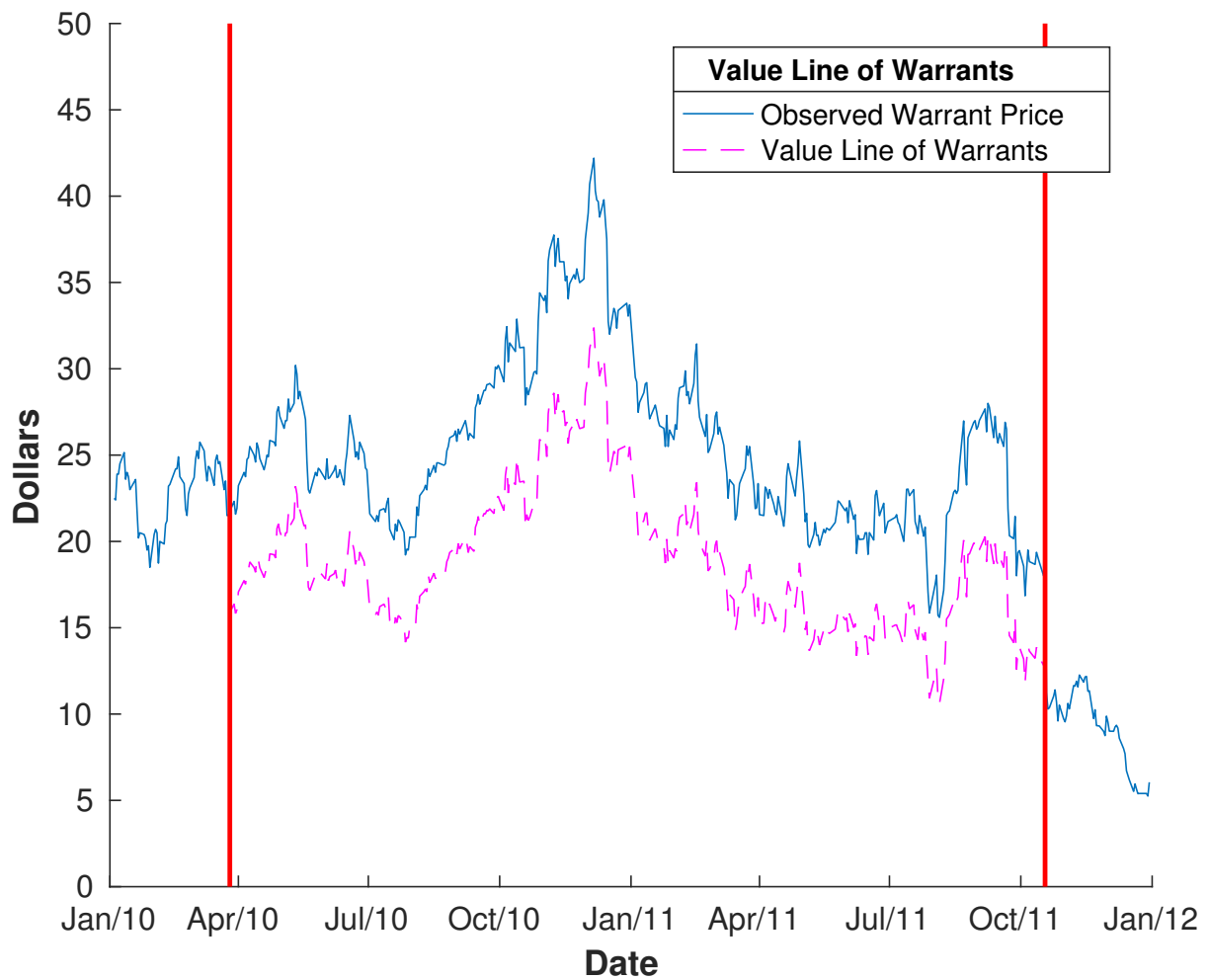


Figure 2.20: Value line of warrants - The period between the two vertical lines in is the class period. The blue solid line shows the historical warrant price of AEM from January 2010 to December 2011. The warrant price value line, represented by the magenta dashed line, is calculated by the methodology discussed in Section 2.5

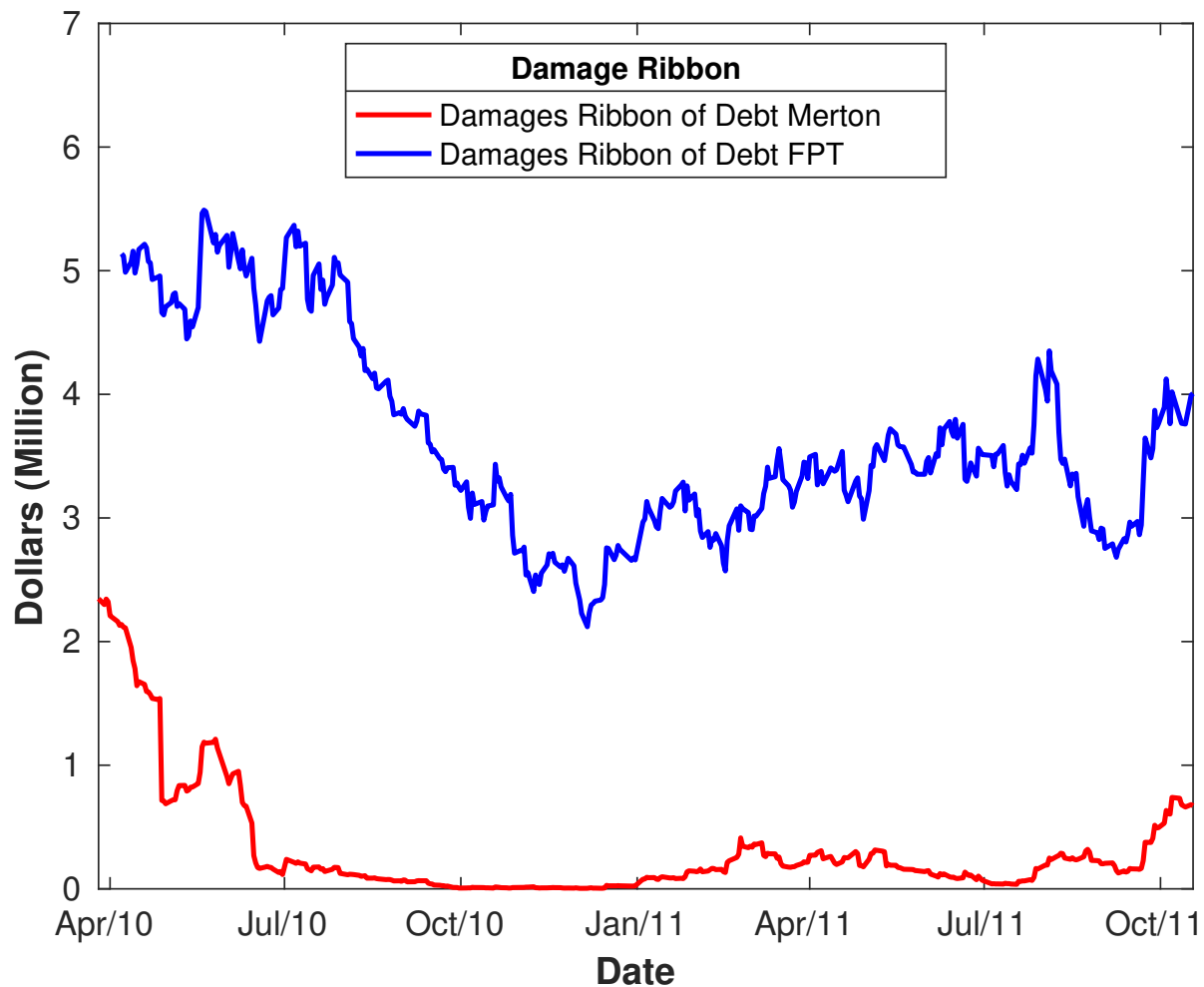


Figure 2.21: Damages Ribbon of Debt - The red line and blue line show the damages ribbons of debt computed using the Merton model and the FPT model, respectively, during the period from April 7, 2010 to October 18, 2011.

Bibliography

- [1] Isabel Abínzano and Javier F. Navas. Pricing levered warrants with dilution using observable variables. *Quantitative Finance*, 13(8):1199–1209, 2013.
- [2] Agnico-Eagle Mines Limited. Agnico-Eagle Mines Limited, 2010 Annual Report. <http://www.agnicoeagle.com/>, 2011.
- [3] Ray Ball and Philip Brown. An empirical evaluation of accounting income numbers. *Journal of Accounting Research*, 6(2):159–178, 1968.
- [4] Carson C. Block. Muddy waters initiating coverage on tre.to, otc:snoff-strong sell, 2011.
- [5] Bradford Cornell and R Gregory Morgan. Using finance theory to measure damages in fraud on the market cases. *UCLA Law Review*, 37:883, 1989.
- [6] M. Crouhy and D. Galai. Common errors in the valuation of warrants and options on firms with warrants. *Financial Analysts Journal*, 47(5):89–90, 1991.
- [7] Michel Crouhy and Dan Galai. The interaction between the financial and investment decisions of the firm: The case of issuing warrants in a levered firm. *Journal of Banking & Finance*, 18(5):861–880, 1994.
- [8] Darrell Duffie and David Lando. Term structures of credit spreads with incomplete accounting information. *Econometrica*, 69(3):633–664, 2001.
- [9] Edward A Dyl. Estimating economic damages in class action securities fraud litigation. *Journal of Forensic Economics*, 12(1):1–11, 1999.
- [10] Young Ho Eom, Jean Helwege, and Jing-Zhi Huang. Structural models of corporate bond pricing: An empirical analysis. *The Review of Financial Studies*, 17(2):499–544, 2004.
- [11] Sharpe William F. Capital asset prices: A theory of market equilibrium under conditions of risk*. *The Journal of Finance*, 19(3):425–442.
- [12] Eugene Fama, Lawrence Fisher, Michael Jensen, and Richard Roll. The adjustment of stock prices to new information. *International Economic Review*, 10, 1969.
- [13] Eugene F. Fama. Efficient capital markets: A review of theory and empirical work*. *The Journal of Finance*, 25(2):383–417, 1970.

- [14] Eugene F. Fama. Efficient capital markets: II. *The Journal of Finance*, 46(5):1575–1617, 1991.
- [15] Eugene F. Fama and Kenneth R. French. Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics*, 33(1):3 – 56, 1993.
- [16] Jared Tobin Finkelstein. Rule 10b-5 damage computation: Application of financial theory to determine new economic loss. *Fordham Law Review*, 51:838, 1982.
- [17] Daniel R Fischel. Use of modern finance theory in securities fraud cases involving actively traded securities. *The Business Lawyer*, pages 1–20, 1982.
- [18] Dan Galai and Meir I. Schneller. Pricing of warrants and the value of the firm. *The Journal of Finance*, 33(5):1333–1342, 1978.
- [19] S. Hauser and B. Lauterbach. The relative performance of five alternative warrant pricing models. *Financial Analysts Journal*, pages 55–61, 1997.
- [20] Lintner John. Security prices, risk, and maximal gains from diversification*. *The Journal of Finance*, 20(4):587–615.
- [21] E. Philip Jones, Scott P. Mason, and Eric Rosenfeld. Contingent claims analysis of corporate capital structures: an empirical investigation. *The Journal of Finance*, 39(3):611–625, 1984.
- [22] Beni Lauterbach and Paul Schulzt. Pricing warrants: An empirical study of the Black-Scholes model and its alternatives. *The Journal of Finance*, 45(4):1181–1209, 1990.
- [23] Philip J Leas. The measure of damages in rule 10b-5 cases involving actively traded securities. *Stanford Law Review*, pages 371–398, 1974.
- [24] John Lintner. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. In *Stochastic Optimization Models in Finance*, pages 131–155. Elsevier, 1975.
- [25] Robert C Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.
- [26] Jan Mossin. Equilibrium in a capital asset market. *Econometrica*, 34(4):768–783, 1966.
- [27] James M. Patell and Mark A. Wolfson. The intraday speed of adjustment of stock prices to earnings and dividend announcements. *Journal of Financial Economics*, 13(2):223–252, 1984.
- [28] David Tabak. Risk disclosures and damages measurement in securities fraud cases. *Securities Reform Act Litigation Reporter*, 2006.
- [29] David Tabak and Frederick Dunbar. Materiality and magnitude: event studies in the courtroom. *Litigation services handbook: the role of the financial expert* New York: John Wiley & Sons, 2001.

- [30] Andrey D Ukhov. Warrant pricing using observable variables. *Journal of Financial Research*, 27(3):329–339, 2004.
- [31] Xinghua Zhou and R. Mark Reesor. Misrepresentation and capital structure: Quantifying the impact on corporate debt value. *Journal of Corporate Finance*, 34:293 – 310, 2015.

Chapter 3

Misrepresentation and Capital Structure: First Passage Time Framework

3.1 Introduction

Typical examples of misrepresentation include the overstatement of earnings, the failure to properly disclose the risks of potential liabilities, and accounting irregularities. In most of the cases, misrepresentation results in an overstatement of firm value. For firms whose shares trade in an efficient secondary market, the share price of the firm quickly drops when the misrepresentation is revealed. Using well-established econometric methods, the share price drop is used to assess potential damages to investors who transacted in the share during the class period — the time between the start of the misrepresentation and its revelation.¹ Firms use many vehicles to finance their operations including common and preferred shares, warrants, various debt instruments and employee stock options. The instruments used determine a firm's capital structure and the firm value equals the total value of the component instruments in the capital structure and the expected bankruptcy costs. Fraud affects the value of the entire firm, not just the value of equity. Therefore, assessing damages only due to shareholders can lead to a significant understatement of the losses incurred by investors across all of the firm's issued securities.

In this paper we use structural models to measure the impact of misrepresentation on the value of a firm's debt. Using a relationship between equity and firm value we show how observable equity information can be used to determine firm value and hence the value of debt in the capital structure. Thus the effect of misrepresentation on firm value and debt value can be measured from the observable drop in share price. This leads to a proposed methodology for debt damages assessment consistent with the standard method for assessing damages to equity investors. Furthermore, trades involving corporate bonds happen much less frequently than for common shares and hence corporate debt markets are typically less efficient than equity markets. For example, during the class period there could either be i) no trades involving that firm's debt (i.e., no information about bond value change); or ii) very few trades in an inefficient market (i.e., potentially unreliable information about bond value change). In such situations,

¹In this paper we use the terms misrepresentation and fraud interchangeably, although we recognize there are important legal distinctions between them.

the modelling framework presented here allows one to compute bond value changes consistent with the observed share price change. Additionally in some jurisdictions (e.g., Province of Ontario, Canada) damages awarded are net of any hedge or risk-limitation transaction. Since corporate securities such as bonds, stocks and warrants are often held in portfolios for hedging purposes (e.g., short equity position to hedge a long bond position), measuring the effect of misrepresentation on all of the firm's issuances is essential to accurately computed damages awards. Hence, our findings and proposed methodology have important consequences for damages assessment and allocation of settlement awards in securities class actions.²

Our study shows that misrepresentation has a significant impact on the value of all debt components in the capital structure. We find that the debt for higher-leverage firms is more sensitive to the misrepresentation impact than for lower-leverage firms and junior debt is more affected by fraud than senior debt. Furthermore, the First Passage Time model (the Black and Cox [5] model or the one-factor Longstaff and Schwartz [37] model³) produces security values that are more sensitive to fraud size than the canonical Merton model [39]. This result is unsurprising, since it is well known that the relatively inflexible Merton model generates credit spreads that are too small to reflect actual credit spreads (i.e., it overprices bonds), hence leaving little opportunity for significant changes in bond prices due to fraud-induced drops in share price.

In addition to our main findings, we explicitly discuss bankruptcy costs in the First Passage Time model. Furthermore, we are able to reduce a system of two nonlinear equations, used to connect the unobservable firm value and firm value volatility to observable equity value and equity volatility, into one equation. This technique improves the ability to solve the non-linear system, leading to a more efficient method for connect observable equity value to firm value and it provides explicit justification for use of the maximum likelihood method [16] for the capital structure models considered here.

The paper is organised as follows. The rest of Section 3.1 discusses a standard econometric method for assessing the impact of fraud on share price and also introduces the “constant percentage change” model. Section 3.2 discusses the First Passage Time model with bankruptcy costs. Section 3.3 gives the connection between the unobservable firm value and the observable share price, critical to measuring the impact of fraud on debt value. In Section 3.4 we investigate the effect of fraud on debt value and propose a methodology to compute the damages for a bond. A summary and concluding discussion is given in Section 3.5.

3.1.1 The Value Line and Constant Percentage Change Model

An efficient market⁴ is one in which publicly-available information and signals are quickly evaluated and reflected in market prices. Stock markets such as NYSE and TSX are considered efficient markets. There are many different ways to measure market efficiency available in the literature [3], [23], [43], [25]. Both firm-specific (idiosyncratic) and general market (systematic) information affect a firm's stock price. Under this presumption the stock price is viewed

²Note that this paper concerns only the impact of fraud on the value of firm-issued securities, not on the value of third-party issued securities, such as exchange-traded single-name equity options.

³We refer to this model as the First Passage Time model in the rest of the paper.

⁴Economists typically refer to three forms of financial market efficiency — weak, semi-strong and strong. The reader is referred to [24] for further discussion of the Efficient Market Hypothesis.

as a function of both pieces of information.

A two-factor linear model is the basic financial econometric model used to estimate a security's true value and from which damages estimates are computed. This model specifies the security's expected return as a linear function of the return on the whole market and the return on the industry sector. Specifically,

$$R_t = \alpha + \beta_1 R_{Mt} + \beta_2 R_{It} + e_t, \quad (3.1)$$

where R_t , R_{Mt} , and R_{It} are the time- t stock, market, and industry returns, respectively, and e_t is the residual value (assumed to be independent random variables with mean zero and constant variance). Using Equation 3.1 the expected return of the stock at time t is

$$E[R_t] = \alpha + \beta_1 R_{Mt} + \beta_2 R_{It}. \quad (3.2)$$

Regression analysis with historical data is performed to estimate α , β_1 , and β_2 . Suppose that τ_b is the start date of the fraud and τ_e is the date the misrepresentation is revealed, so that $[\tau_b, \tau_e]$ defines the class period. Using this econometric model, a value line representing the true stock value (i.e., the path the stock price would have followed in the absence of misrepresentation or omission during the class period) can easily be constructed [11]. This method constructs a series of daily returns R_{Ct} in the following way

- if no fraud-related information is disclosed, set the return equal to the actual return on the security;
- if fraud-related information is disclosed or leaked into the market, set the return for these days equal to the expected return calculated using Equation 3.2 and the estimates for α, β_1, β_2 .

This series of daily returns is then used to construct the value line by

$$\tilde{S}_{t-1} = \tilde{S}_t / (1 + R_{Ct-1}), \quad (3.3)$$

where \tilde{S}_t is the true value of stock at time t .

Alternative ways of constructing the value line are the constant percentage method and the constant dollar method. The constant percentage method assumes that the misrepresentation has a proportional effect on the stock price, i.e.,

$$\tilde{S}_t = \delta S_t, \quad (3.4)$$

for $t \in [\tau_b, \tau_e]$, where δ is the constant percentage change during the class period. Typically, δ is determined by the size of the stock price movement controlled for changes in the market and industry on the date the fraud is revealed. This is used with the model and method discussed above to construct the stock's value line. The constant dollar method assumes a fixed dollar change in stock price during the class period; and for reasons of brevity, it is not used in this paper.

Much work has been done on the methodology for estimating damages in securities fraud cases. A non-exhaustive list is [32], [27], [26], [11], [19], [49] [50] and [48] which provide variants on the methodology given here. What is missing in these works, however, is the impact of fraud on the value of other securities in the capital structure. In this paper our proposed method assesses the impact of fraud on the value of debts, given the easily measured and observed impact on the share price.

3.2 The First Passage Time Model with Bankruptcy Costs

In this paper, we measure damages for debts using the one-factor Longstaff-Schwartz [37] model, which is essentially the Black-Cox [5] model with a flat default boundary.⁵ There is a large literature on structural models that focus on a variety of economic considerations, such as stochastic interest rates, exogenous default, endogenous default and stationary leverage ratios. The purpose of this paper is to propose a methodology for computing damages for secondary market securities class actions that uses a capital structure model. The First Passage Time model we selected to illustrate our methodology is simple to implement and allows for a variety of debt instruments (seniority, coupon, maturity) and bankruptcy costs. Other structural models that accommodate different features could be used in a straightforward manner. Here, we briefly discuss model assumptions and derive the closed-form valuation expressions for equity, debt, and bankruptcy costs.

3.2.1 Valuation of Debt

We assume a flat term structure in which the risk free rate r is constant. Let V^F be the total value of the assets of the firm. Assume that the assets follow the dynamic

$$dV_t^F = (r + \lambda^v \sigma^F) V_t^F dt + \sigma^F V_t^F d\hat{W}_t, \quad (3.5)$$

where λ^v , r , and σ^F are constant and \hat{W} is a Brownian motion under the real world measure \mathcal{P} . We assume perfect, frictionless markets in which no arbitrage opportunity exists. We assume that default is exogenous and that there is a threshold value K ; when the firm value V^F reaches the constant boundary K , the firm enters financial distress and simultaneously defaults on all of its obligations. The time- t price of T -maturity risky debt with face value one is given by

$$P(V_t^F, \sigma^F, r, w, K, t, T) = e^{-r(T-t)} E_Q[1 - wI_{\tau \leq T}] = e^{-r(T-t)} (1 - wQ(\tau \leq T)), \quad (3.6)$$

where w is the loss given default, τ is the first passage time of the firm value V^F to the boundary K , I_A is an indicator function of the event A and $Q(\tau \leq T)$ is the probability of the event $[\tau \leq T]$ under the risk-neutral measure. The time- t conditional distribution of the first passage time, τ , is

$$\begin{aligned} Q(\tau \leq T) = & \Phi\left(\frac{-\log(V^F/K) - r(T-t) + 0.5\sigma^{F^2}(T-t)}{\sigma^F \sqrt{T-t}}\right) \\ & + \exp\left(\frac{-2\log(V^F/K)(r - 0.5\sigma^{F^2})}{\sigma^{F^2}}\right) \\ & \times \Phi\left(\frac{-\log(V^F/K) + r(T-t) - 0.5\sigma^{F^2}(T-t)}{\sigma^F \sqrt{T-t}}\right), \end{aligned} \quad (3.7)$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.⁶

⁵Predescu [44] use a similar modelling set up (without bankruptcy costs) to study the performance of structural models using CDS spreads.

⁶Please refer to [28] for the derivation of the first passage time distribution.

The time- t value of a T -maturity zero coupon bond with face value F is $F \times P(V_t^F, \sigma^F, r, w, K, t, T)$. For debt with coupon payments, we evaluate it using the “portfolio of zeros” approach [37], i.e., treat each coupon payment as a “mini” zero-coupon bond and compute the coupon bond value as the sum of the mini zero-coupon bond values. As noted in Eom, Helwege and Huang [21] this approach fails to take into account the dependence of the mini zero-coupon bonds. Incorporating the dependence of these coupon payments could improve model performance. Consider a risky coupon bond with face value F and N_C remaining coupon payments of C . Let TC_i be the i th-coupon payment date from a given time t , and TC_{N_C} be the last coupon payment date, which is T . The value of the risky coupon bond B_t at time t is the value of the corresponding portfolio of zeros, i.e.,

$$B_t = C \sum_{i=1}^{N_C-1} P(V_t^F, \sigma^F, r, w, K, t, TC_i) + (F + C)P(V_t^F, \sigma^F, r, w, K, t, T). \quad (3.8)$$

For a firm with N^D different bonds outstanding, the present value of the firm’s total liabilities D_t is the sum of all the bond prices, which is given by

$$D_t = \sum_{j=1}^{N^D} B(V_t^F, \sigma^F, r, w^j, K, t, T^j, C^j, F^j), \quad (3.9)$$

where w^j , T^j and C^j are the parameter inputs corresponding to the j th-bond and B is given by equation (3.8). Note that the loss given default w^j may be different across bonds and that w^j is the same for the principle F^j and the coupon payment C^j (see [29]). Research on recovery rates of debt has been studied by many authors (see [2] and [42]). Ou, Chiu, and Metz [42] found the value weighted average loss given default of senior secured debt w^s to be 50.9% while the loss given default of a junior subordinated debt w^j to be 82.9%. For the case with only one type of debt outstanding, we set w as 48.67% following [29] and [21].

Let $PV(D)$ be the present value of the debts’ face value discounted at the risk free rate. That is,

$$PV(D) = \sum_{j=1}^{N^D} \left((C^j + F^j)e^{-r(T^j-t)} + C^j \sum_{i=1}^{N_C^j-1} e^{-r(TC_i^j-t)} \right), \quad (3.10)$$

and let \bar{T} be the weighted average maturity date of all debts, namely,

$$\bar{T} - t = \sum_{j=1}^{N^D} \left((T^j - t) \frac{(C^j + F^j)e^{-r(T^j-t)}}{PV(D)} + \sum_{i=1}^{N_C^j-1} (TC_i^j - t) \frac{C^j e^{-r(TC_i^j-t)}}{PV(D)} \right). \quad (3.11)$$

The default boundary K can be set equal to the time- \bar{T} value of all future liability payments. For example, consider a firm with two bonds, a and b , outstanding. The face values are F^a and F^b and the maturities are T^a and T^b , respectively. The default boundary, K , of this firm is set equal to $F^a e^{r(\bar{T}-T^a)} + F^b e^{r(\bar{T}-T^b)}$.

3.2.2 Barrier Option Framework for Equity Value

The equity value is given as the value of a European down-and-out call option written on firm value V^F . Similar approaches to valuing equity can be found explicitly in [7] and [44]

and implicitly in [30]. The price of a European down-and-out call option on the firm value C_{DO} with barrier K_B and strike price K_S is $C_{DO}(V_t^F, \sigma^F, r, K_B, K_S, t, T)$, whose exact formula is given in B.1.⁷ Under the First Passage Time model framework, a bond holder would suffer no loss provided the firm value V_t^F never reaches the default boundary K prior to maturity T . In the case of a single zero-coupon bond, the default boundary K is at least equal to the face value of the bond in order to guarantee sufficient asset value to pay off the debt at T . For a general debt structure the time- t equity value EQ_t is given by

$$EQ_t = V_t^F \Phi(d_1) - Ke^{-r(\bar{T}-t)} \Phi(d_2) - V_t^F (K/V_t^F)^{2\lambda} \Phi(d_3) + Ke^{-r(\bar{T}-t)} (K/V_t^F)^{2\lambda-2} \Phi(d_4), \quad (3.12)$$

where

$$\lambda = (r + \sigma^{F^2}/2)/\sigma^{F^2} \quad (3.13)$$

$$d_1 = (\log(V_t^F/K) + (r + \sigma^{F^2}/2)(\bar{T} - t))/(\sigma^F \sqrt{\bar{T} - t}), \quad (3.14)$$

$$d_2 = d_1 - \sigma^F \sqrt{\bar{T} - t}, \quad (3.15)$$

$$d_3 = (\log(K/V_t^F) + (r + \sigma^{F^2}/2)(\bar{T} - t))/(\sigma^F \sqrt{\bar{T} - t}), \quad (3.16)$$

and

$$d_4 = d_3 - \sigma^F \sqrt{\bar{T} - t}. \quad (3.17)$$

Given the number of outstanding shares, N , of the firm, the stock price of the firm is the equity value divided by N . In Proposition 3.3.1 we show that EQ_t defined by equation (3.12) is a one-to-one function of V_t^F (see B.2). This is used in the connection between equity value and firm value providing a more efficient implementation (Section 3.3.1) and explicit justification for use of the maximum likelihood method (Section 3.3.2).

3.2.3 Bankruptcy Costs

In the case of a single zero-coupon bond outstanding, when the firm value V^F reaches the default boundary K at time $\tau < T$, the equity value becomes zero, and hence the realized bankruptcy cost BC_τ is the difference between the default boundary K and the value of recovered risk-free bond $e^{-r(T-\tau)}(1-w)F$.⁸ Before default happens, firm value is the sum of the debt and equity values and the expected present value of bankruptcy cost. The expected present value of bankruptcy costs at time t is

$$BC_t = V_t^F - D_t - EQ_t, \quad (3.18)$$

where D_t is the time- t debt value. BC_t defined in Equation (C.4) is a decreasing, convex function of the firm value V_t^F . It has the same properties as the bankruptcy cost defined in [33] and [35], i.e., BC_t satisfies the boundary conditions⁹

$$\text{at } V_t^F = K, \quad BC_t = K - e^{-r(T-t)}(1-w)F, \text{ and} \quad (3.19)$$

$$\text{as } V_t^F \rightarrow \infty, \quad BC_t \rightarrow 0. \quad (3.20)$$

⁷Barrier option valuation can be found in [46] and [40].

⁸In the general debt structure case the recovered risk-free debt has value $(1-\bar{w})PV(D)$, where \bar{w} is the resulting loss given default for all outstanding debt given that bonds with different seniority have different recovery rates.

⁹In the general debt structure case the first condition becomes $BC_t = K - (1-\bar{w})PV(D)$ when $V_t^F = K$.

It is easy to see that the boundary condition (C.5) holds. Boundary condition (C.6) also holds: when the firm value V_t^F becomes very large, the debt value D_t approaches to $e^{-r(T-t)}F$, which is the value of risk-free debt with the same maturity and face value, and the equity value EQ_t approaches to $V_t^F - Ke^{-r(T-t)}$, which is the upper bound for a European call option price under the Black-Scholes framework, and hence the present value of the bankruptcy costs BC_t approaches to zero, because $K = F$.¹⁰ A numerical example in the case of a single zero-coupon bond is illustrated in Figure 3.1.

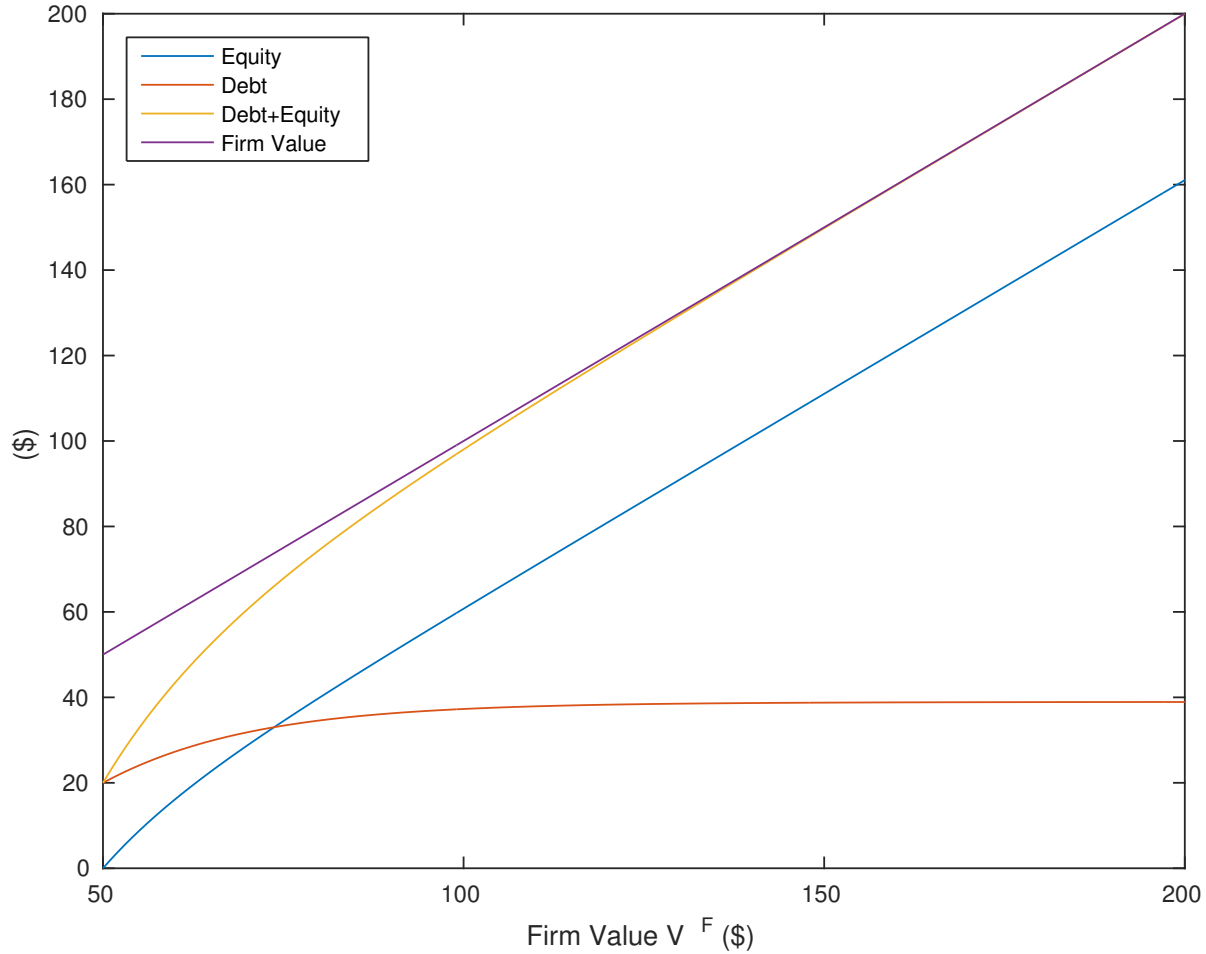


Figure 3.1: Firm Value and Capital Structure - Firm value V_t^F versus equity value EQ_t , debt value D_t and the sum of EQ_t and D_t using the model in Section 3.2. The bankruptcy costs BC_t are the difference between V_t^F and $D_t + EQ_t$. Parameters are: $\sigma^F = 21\%$, $r = 5\%$, $T - t = 5$, $K = 50$, and $w = 48.67\%$ (parameter values r and w are chosen following Huang and Huang[29]).

In equation (3.12) we assume that the barrier K_B and the strike price K_S of the down-and-

¹⁰In the general debt structure case, the bankruptcy costs BC_t also satisfies the second boundary condition. When the firm value V_t^F goes to infinity, the probability of default becomes zero, and hence the debt value D_t approaches to $PV(D)$. At the same time, the value of equity EQ_t , which is given by equation (3.12), approaches $V_t^F - Ke^{-r(T-t)}$ as V_t^F goes to infinity. By the definition of K , $Ke^{-r(T-t)}$ equals to $PV(D)$. Hence BC_t defined by equation (C.4) approaches $V_t^F - PV(D) - (V_t^F - PV(D))$, which is zero.

out call option are both equal to the time- \bar{T} value of future liability payments. It is possible to relax this assumption in the case when a firm issues multiple bonds. Setting the default boundary K equal to the time- \bar{T} value of future liability payments results in high bankruptcy costs when the loss given default w is high (see discussion in [29]): in our case, the 48.67% loss given default w implies that the bankruptcy costs are around 50%, with the precise value depending on the default time. However, empirical studies show that bankruptcy costs are usually lower than 50%, for example, Davydenko, Strebulaev, and Zhao[15] found the average bankruptcy costs to be 21.7% of the market value of the firm's assets. To lower the bankruptcy costs, BC , one can set the strike price K_S of the down-and-out call equal to the time- \bar{T} value of future liability payments, and set the barrier K_B to be the default boundary that is less than K_S . Under this setup, the value of equity is given by equation (B.1) and the bankruptcy costs, BC_t , defined by equation (C.4) still satisfy the boundary conditions (C.5) and (C.6) with K replaced by K_B . Evidence [8] on loan covenants suggests that the default boundary is much lower than the nominal debt value. The study [8] reports an average initial threshold of maximum debt to total net worth equal to 1.81. Discussion of how to set the default boundary K can be found in [12], [21], [44], [34], [30], [29], and [14]. In the following discussion of this paper, we consider the case in which both K_B and K_S are equal to the time- \bar{T} value of future liability payments.

3.3 Connection between Equity Value and Firm Value

In this section we discuss the relation between the equity value and firm value for the capital structure model discussed in Section 3.2. Qualitatively, it is well known that debt values increase with equity value, due to a decrease in the likelihood of default. Thus as equity value increases, debt values increase, and hence leveraged firm value increases. Conversely, debt and leveraged firm values decrease with decreases in equity value.

Moreover, it is easily seen that the equity volatility and the firm value volatility cannot be constant at the same time. A firm is financed by a combination of equity, debt and other securities in its capital structure. Hence firm value volatility is a combination of the volatility of equity, debt and other capital structure components. If a firm's equity value changes, for example, the equity to firm value ratio changes and hence so does the contribution of equity volatility to firm value volatility.

In this section, we provide two methods to connect the observable equity value EQ_t and its volatility σ^E to the unobservable firm value V_t^F and its volatility σ^F . We discuss the traditional method and present a technique to improve computational performance in Section 3.3.1. We briefly discuss the maximum likelihood estimation (MLE) method in Section 3.3.2 with the mathematical details given in B.4.

3.3.1 The Traditional Method

Given the equity value EQ_t and its volatility σ^E , the firm value V_t^F and σ^F can be found by solving a non-linear system as discussed in [45] and [31]. We refer to this method as the traditional method. Under the risk neutral measure, the firm value dynamics are

$$dV_t^F = rV_t^F dt + \sigma^F V_t^F dW_t, \quad (3.21)$$

where W is a Brownian motion. Since EQ_t is a function of V_t^F , it can be shown by Itô's Lemma that the equity volatility is

$$\sigma^E = \sigma^F \frac{\partial EQ_t}{\partial V_t^F} \frac{V_t^F}{EQ_t}, \quad (3.22)$$

where $\partial EQ_t / \partial V_t^F$ is given in equation (B.10).

Combining equation (3.22) with the valuation expression of equity value (see equation (3.12)) constructs a system of two non-linear equations. Thus given EQ_t and σ^E , the value of V_t^F and σ^F can be found by simultaneously solving the equations,

$$EQ_t = V_t^F \Phi(d_1) - Ke^{-r(\bar{T}-t)} \Phi(d_2) - V_t^F (K/V_t^F)^{2\lambda} \Phi(d_3) + Ke^{-r(\bar{T}-t)} (K/V_t^F)^{2\lambda-2} \Phi(d_4), \quad (3.23)$$

$$\sigma^E = \frac{\sigma^F V_t^F}{EQ_t} \left[\Phi(d_1) + (2r/\sigma^{F^2})(K/V_t^F)^{2\lambda} \Phi(d_3) + (1 - 2r/\sigma^{F^2})(K/V_t^F)^{2\lambda-1} e^{-r(\bar{T}-t)} \Phi(d_4) \right]. \quad (3.24)$$

Proposition 3.3.1 *If parameters r , σ^F , K_S , K_B , t , and T are given and satisfy $r \geq 0$, $\sigma^F > 0$, $K_S \geq 0$, $K_B \geq 0$, and $T > t$, respectively, the value of equity EQ_t given by equation (3.12), (B.1) or (B.5) is a one-to-one function of the firm value V_t^F , for $V_t^F \in [K_B, \infty)$.*

Proof of the proposition is shown in B.2. The proposition explicitly shows that the equity value EQ_t in (3.23) is a one-to-one function of V_t^F . In other words, given parameters r , K , T , t , and EQ_t , for any firm volatility $\sigma^F \in (0, \infty)$, there is only one firm value V_t^F satisfying equation (3.23). For any given equity value EQ_t , we can define a function v_{EQ_t} such that $V_t^F = v_{EQ_t}(\sigma^F)$ satisfies equation (3.23). Since the firm value V_t^F is completely determined by the firm volatility σ^F , instead of simultaneously solving the non-linear system (3.23) and (3.24), we only need to solve one equation

$$\sigma^E = \sigma^F \frac{\partial EQ_t}{\partial V_t^F} \left(\sigma^F, v_{EQ_t}(\sigma^F) \right) \frac{v_{EQ_t}(\sigma^F)}{EQ_t} \quad (3.25)$$

for σ^F .

Theorem 3.3.2 *Given equity value EQ_t and its volatility σ^E , solving the non-linear system (3.23) and (3.24) for V_t^F and σ^F is equivalent to solving the following equation for σ^F :*

$$\begin{aligned} \sigma^E = \frac{\sigma^F v_{EQ_t}(\sigma^F)}{EQ_t} & \left(\Phi(\hat{d}_1) + (2r/\sigma^{F^2})(K/v_{EQ_t}(\sigma^F))^{2\lambda} \Phi(\hat{d}_3) \right. \\ & \left. + (1 - 2r/\sigma^{F^2})(K/v_{EQ_t}(\sigma^F))^{2\lambda-1} e^{-r(\bar{T}-t)} \Phi(\hat{d}_4) \right), \end{aligned} \quad (3.26)$$

where

$$\hat{d}_1 = (\log(v_{EQ_t}(\sigma^F)/K) + (r + \sigma^{F^2}/2)(T - t))/(\sigma^F \sqrt{T - t}), \quad (3.27)$$

$$\hat{d}_3 = (\log(K/v_{EQ_t}(\sigma^F)) + (r + \sigma^{F^2}/2)(T - t))/(\sigma^F \sqrt{T - t}), \quad (3.28)$$

$$\hat{d}_4 = \hat{d}_3 - \sigma^F \sqrt{T - t}, \quad (3.29)$$

and the function v_{EQ_t} from $(0, \infty)$ to $[K, \infty)$ is defined for any given EQ_t , such that $V_t^F = v_{EQ_t}(\sigma^F)$ satisfies equation (3.23). The firm value V_t^F is $V_t^F = v_{EQ_t}(\sigma^F)$, where σ^F comes from solving (3.26).

The proof Theorem 3.3.2 is given in B.3. The practical implication of Theorem 3.3.2 lies in the fact that when solving the non-linear system simultaneously using numerical method, it is possible to get different solutions depending on the initial guess¹¹. Hence it can be necessary to manually select the most reasonable solution from multiple solutions (see [36]). By reducing the non-linear system into one equation, the issue of multiple solutions can be avoided. Even though Theorem 3.3.2 is derived under the model in Section 3.2, it is easily extended to other structural models admitting a one-to-one relationship between equity and firm value such as the Merton model [39].¹²

3.3.2 Maximum Likelihood Estimation (MLE)

The traditional method of solving for V_t^F and σ^F has a straightforward implementation and is widely applied in academic studies. This section provides a popular alternative, the MLE method, to estimate the firm value and firm volatility from observable equity information. The method has also been extended to incorporate market information about default probabilities from credit default swaps [44].

Recent studies (see [16], [17], [22], and [18]) show that the traditional method is misspecified: structural models imply that the equity volatility σ^E is a function of V_t^F and σ^F , but typically σ^E is estimated assuming that it is constant. Using simulation, Ericsson and Reneby[22] show that the MLE method proposed by Duan[16] is better than the traditional method in estimating the firm value V_t^F and firm volatility σ^F . An empirical study by Li and Wong[36] shows that the MLE method substantially improves the performance of structural models in pricing corporate bonds.

However, when using the MLE method, the firm volatility is assumed constant during the estimation period. If the leverage ratio of the firm changes during this period, i.e., bonds are expired/issued, it might be necessary to divide the time period into sub-periods and estimate the firm volatility for each sub-period. This issue does not arise for the traditional method because the firm volatility changes daily and is calculated based on the daily equity volatility. In addition, the MLE method assumes that the market price of risk, λ^V , is constant over the estimation period (a specific and restrictive assumption). On the other hand, the traditional method does not require such a restrictive assumption. The results in Huang and Zhou [30] shows that the traditional method still works when used in combination with Generalized Method of Moments estimation.

Let $EQ^{TS} \equiv \{EQ_i^{ts} : i = 1, 2, \dots, n\}$ be a daily time series of equity value, where i is the time index. Using MLE, the firm volatility σ^F can be estimated from EQ^{TS} . Given the maximum likelihood estimator of firm volatility $\hat{\sigma}^F$, the maximum likelihood estimator of firm value \hat{V}_t^F is just $v_{EQ_t}(\hat{\sigma}^F)$. The MLE method requires that the equity value be a one-to-one function of firm value V_t^F given the other parameter values (see equation (B.16)). Proposition 3.3.1 provides the justification for using the MLE method with the structural model in Section 3.2. Details of the implementation can be found in B.4. Another popular approach to estimate

¹¹For some initial guesses, numerical method may fail to converge to the true solution of the non-linear system; it gives suboptimal solutions in these cases as in finding a local optimum rather than a global optimum.

¹²For some capital structure models that include warrants (e.g. [13]) there is not a one-to-one relationship between firm and equity values.

the firm value and firm value volatility is the iterative (KMV) approach [12] and [38], which is equivalent to the MLE approach under the Merton framework [18].

3.4 Misrepresentation and Debt Value

In this section we study the relation between fraud and debt value based on the connection between the equity value and firm value discussed in the previous section. When the underlying stock price is inflated/deflated by misrepresentation or omission, the debt value will also be inflated/deflated. We propose a methodology to quantify the effect of misrepresentation on debt value and use it to construct the damages ribbon for debt. The damages ribbon is the difference between the debt value computed using the observed share price and the debt value computed using the value line for the share price (see Section 3.1.1). An important advantage of using the econometric model (see Section 3.1.1) to compute stock value line is that it measures the effect of fraud related information on the stock price and filters out the price movement due to non-fraud related information. As such, the fraud impact is measured using observed information from the efficient equity market. Our methodology takes the fraud impact signal observed from the equity market and, using a capital structure model, translates this to the impact of fraud on debt value thus providing damages assessments that are consistent across all capital structure components. In Section 3.4.1, we discuss the effect of fraud on debt value with numerical examples. We demonstrate the proposed method in a case study by constructing the bond damages ribbon for the recent Agnico Eagle Mines securities class action case in Section 3.4.2.

3.4.1 Effect of Fraud on Debt Value

In this section we show how fraud affects the debt value based on the model of Section 3.2. When implementing the traditional method to measure the effect of fraud, we fix the firm value volatility and allow the stock price volatility to vary; this is consistent with the fundamental principle of the MLE method. We define the relative price

$$\delta = \frac{\tilde{S}_t}{S_t} \quad (3.30)$$

and use δ as a measure of the fraud size. Values of δ greater than 1 indicate that the fraud depressed the share price (not common) and values of δ less than one correspond to fraud inflating the share price. If $\delta = 1$ then $\tilde{S}_t = S_t$ and there is no misrepresentation.

We plot the fraud size δ versus the value of junior and senior debt (normalized to face value) using the First Passage Time (FPT) model and the Subordinated Debt (SD) model (see B.5) in Figures 3.2 and 3.3, respectively. In both models, we set the initial equity value as 100, and use the quasi-leverage ratio to compute the corresponding default boundary/face value. The quasi-leverage ratio L is defined as the quasi market value of debt to the quasi asset value, i.e., $L = (e^{-r(T-t)}K)/(EQ_t + e^{-r(T-t)}K)$ and $L = (e^{-r(T-t)}F)/(EQ_t + e^{-r(T-t)}F)$ for the FPT and the SD models, respectively.

Following the empirical study of Schaefer and Strebulaev[47], we set the leverage ratio and equity volatility as 0.10, 0.32, and 0.50 and 25%, 31%, and 42% for AAA, A, and BB credit-

rated firms, respectively. The constant risk-free rate is 5%, the time to maturity of the debts is 5 years and the losses given default are $w^j = 82.9\%$ and $w^s = 50.9\%$, respectively. Given the equity value and corresponding equity volatility, the firm value and firm volatility are computed using the traditional method. The true value of firm is computed using the non-linear system (given in the traditional method) by using the true equity value and the firm volatility. The effect of fraud on debt value can be computed by taking the difference in debt values when $\delta = 1$ and $\delta \neq 1$.

In the FPT model we see that junior debt value is more sensitive to fraud size than senior debt value, with this sensitivity increasing with leverage. With a low-leverage firm ($L = 0.10$), the likelihood of default is so low that the fraud sizes considered here have no effect on the debt values.

Using the SD model that includes both junior and senior debts, we find that the effect of fraud size on debt value increases with leverage and that junior debt value is more sensitive to fraud size than senior debt value (in accordance with our findings with the FPT model). The latter observation is a result of the absolute priority rule in allocating losses, with the junior debt serving as a loss-absorbing cushion for the senior debt. Comparing results between the two models, we see that debt values in the FPT model are more sensitive to fraud size than those from the SD model. This is due to a combination of factors including i) the bankruptcy costs; ii) violation of the absolute priority rule; and iii) the possibility of default prior to bond maturity.

3.4.2 Calculating the Damages Ribbon for Debt

In the previous sections, we have shown that misrepresentation not only inflates/deflates the underlying share price but also inflates/deflates the debt value of the firm during the class period. In Section 3.4.2, we propose a method to construct the debt value line (equivalently the damages ribbon) based on the share price value line. We demonstrate the proposed method with a case study in Section 3.4.2

Methodology for Debt Damages Ribbon

Given the share price value line we propose a method for constructing the debt value line (equivalently the damages ribbon for debt). The true value of a bond on a date i during the class period can be calculated using the following steps:

- 1 Calculate the true value of equity \widetilde{EQ}_i using the share price value line as described in Section 3.1.1;
- 2 Calculate the firm value V_i^F and firm value volatility σ_i^F using
 - a the traditional method and the equity value EQ_i and equity volatility σ_i^E ; or
 - b using the MLE method and EQ^{TS} , which is the time series of equity value from a given sample window;
- 3 Compute the true firm value $\tilde{V}_i^F = v_{EQ_i}(\sigma_i^F)$;

- 4 Calculate the bond values $B(V_i^F, \sigma_i^F)$ and $B(\tilde{V}_i^F, \sigma_i^F)$ based on the firm values V_i^F and \tilde{V}_i^F , respectively, and hence the date- i damages ribbon $R_i^B = B(V_i^F, \sigma_i^F) - B(\tilde{V}_i^F, \sigma_i^F)$.

When calculating the share price value line, the econometric model in Section 3.1.1 measures the effect of fraud related information on the stock price and the effect of non-fraud related information is filtered out by the model. Since the debt value line is calculated based on the share price value line, the debt value line preserves this advantage — the debt damages ribbon reflects the bond value change due to the fraud related information. Additionally, the debt value line/damages ribbon provides a methodology for assessing debt damages that is consistent with the standard method for computing equity damages. This is important in the assessment of damages because some jurisdictions (e.g., Ontario Securities Act Section 138.5(1)) require that damages are calculated taking into account the result of hedging and other risk limitation transactions. Moreover, it is straightforward to implement this methodology as it only requires observable information (e.g., equity value, treasury bond yield and corporate financial statements). A limitation of the methodology is that the structural model used may not be rich enough to sufficiently capture all relevant features of the capital structure (e.g., callable/convertible features of bonds) and the bankruptcy process — this limitation generally arises in any modelling exercise. Additionally other factors such as liquidity, call and conversion features and taxes are also the key determinants of corporate bond prices (see [20], [9], [4], and [29]). By using a structural model to compute the debt damages ribbon, the price change reflects the effect of fraud on bond value only due to changes in credit risk. This approach does not reflect how fraud may influence the other factors (if at all) that determine corporate bond prices. In the following section, the proposed methodology is demonstrated using a recent securities class action case.

The Agnico-Eagle Mines Ltd. Case Study

Agnico-Eagle Mines Ltd. (AEM) is a Canadian based mining company with operations in Canada, Finland, Mexico and the United States. In March 2012, secondary market securities class actions were filed in Ontario and Quebec against AEM and certain of AEM's current and former officers and directors. The actions allege that AEM failed to disclose a water inflow issue at its Goldex mine (in Quebec) during the class period — March 26, 2010 to October 18, 2011 (indicated by the vertical red lines in Figure 3.4). In Figure 3.4, the stock price drops significantly on the fraud disclosure date (October 18, 2011) indicating that the share price was inflated during the class period. The common share value line was calculated using the event study approach as described in Section 3.1.1. The S&P/TSX Composite Total Return Index and the S&P/TSX Gold Total Return Index are used as the market return R_{Mt} and the industry return R_{It} , respectively. Two years of daily observations from January 04, 2010 to December 30, 2011 are used to estimate the parameters in the econometric model. On October 18, 2011, there was a price change of \$10.59 of which \$7.03 was attributed to the revelation of fraud after controlling for market and industry factors.

During the class period the financial statements of AEM indicate that long-term liabilities include two types of long-term debts — bank credit facilities and notes (see [1]). A credit facility is a loan in the form of revolving credit, in which the customer is allowed to borrow/repay funds as needed. Short-term liabilities and the other long-term liabilities (e.g., deferred income and mining tax liabilities) are ignored in our analysis. AEM equity was comprised of

156.7 million common shares, 8.4 million employee stock options, and 8.6 million warrants on March 31, 2010, the start of the class period.¹³ Here we make the simplification that AEM equity is comprised solely of common shares, giving a capital structure for our model consisting of common shares and two types of long-term debt.

On June 22, 2010 (during the class period), AEM amended its credit facilities: the amount available increased to \$1.2 billion and the maturity date extended to June 22, 2014. Details of the interest rate paid on the credit facilities are not available. On April 7, 2010, the company closed a note offering with institutional investors in the U.S. and Canada for a private placement of \$600 million of guaranteed senior unsecured notes due in 2017, 2020 and 2022. The notes had a weighted average maturity of 9.84 years and weighted average yield of 6.59% at issuance. Proceeds from the notes were used to repay amounts owed under the company's then outstanding credit facilities. The credit facilities and the notes rank equally in seniority. Details of the coupon payments and face value of the notes proved difficult to obtain. For this case study, we treat the notes as a single coupon bond with the weighted average maturity and calculate its damages ribbon during the class period. As the outstanding balance of the credit facilities changes dramatically during the class period, we refrain from computing its damages ribbon.

In order to use our methodology, we consider AEM's two types of long-term debt as two coupon bonds. The first bond corresponds to the credit facilities and is assigned a face value of \$69.29 million — the weighted average amount drawn from the credit facilities during the class period. The maturity date of this bond is June 22, 2014. The second bond corresponds to the notes and is given a face value of \$600 million — the total face value of the issued notes. The maturity date of this bond is Feb 8, 2020, the weighted average maturity date of the three notes. Since details of the interest and coupon payments are not available, both bonds are assumed to pay annual coupons of 6.59% (the average yield of the notes), with coupon payments made at the end of each year.

For the other model parameters, the risk-free rate is taken as 3.72%, the average U.S. Treasury long-term composite rate¹⁴ from April 7, 2010 to October 18, 2011. On April 7, 2010, the weighted average time to maturity is 7.4 years, and hence the weighted average maturity date of debts, \bar{T} , is September 02, 2017. The default boundary, K , is \$1070.56 million and is assumed constant during the class period (see Section 3.2.1 for details on calculating the default boundary). The loss given default, w , is 62.6%, which follows Ou, Chiu and Metz [42] who estimated the value-weighted recovery rate for senior unsecured bond as 37.4%¹⁵. Given the time series of equity value¹⁶ from April 7, 2010 to October 18, 2011, the firm value volatility estimate is $\hat{\sigma}^F = 35.24\%$ (using the MLE method). Using $\hat{\sigma}^F$, the time series of equity value, and the value line of equity value, the time series of firm value V_i^F (using the observed share

¹³During the class period, the numbers of common shares and employee stock options changed, while the number of warrants was unchanged. The number of common shares increased from 156.7 million to 169.2 million.

¹⁴The long-term composite rate is the unweighted average of bid yields on all outstanding fixed-coupon bonds neither due nor callable in less than 10 years. Data can be found on the website of the U.S. Department of the Treasury.

¹⁵The recovery rate defined in this chapter is under the recovery of treasury value assumption as in [37], [6], and [10]. However, the Moody's historical recovery rate is the recovery of face value. In the case of zero coupon bond, the difference between the two recoveries is a discount factor multiplier.

¹⁶The daily equity value is the product of the daily close price and the daily current shares outstanding, which are obtained from Bloomberg.

price) and the true firm value \tilde{V}_i^F (using the share price value line) are calculated. From these time series, bond prices and hence the notes damages ribbon are computed according to the methodology in Section 3.4.2. The notes damages ribbon is shown in Figure 3.5 and can be used to assess damages in a securities class action.

3.5 Conclusion

Using an extended Black and Cox capital structure modelling framework and a connection between the observable share price and firm value we connect the impact of an observable fraud-induced share price change on the debt value. Generally, debt for higher-leverage firms is more sensitive to the fraud size than lower-leverage firms and junior debt is more affected by fraud size than senior debt.

This study proposes a methodology to compute damages in securities class actions for investors with debt positions in the fraud-committing company. This work is relevant not only for estimating potential damages, but in the fair allocation of damage awards across holders of the fraudulent firm's securities. The legal requirement that damages due to fraud must be computed net of any hedge or risk limitation transaction underscores the importance of the work presented here. For example, one can hedge a long bond position by shorting the shares of the bond issuer. This methodology allows one to compute fraud-induced share and debt value changes in a consistent manner.

In addition to our main findings, we explicitly discuss bankruptcy costs for the First Passage Time model. Furthermore, we are able to reduce a system of two non-linear equations, used to connect the unobservable firm value and firm value volatility to observable equity value and equity volatility, into one equation. This technique improves the ability to solve the non-linear system.

There are many different avenues for future research. Extending the modelling framework to more general capital structures (e.g., include preferred shares, warrants, and employee stock options) and to incorporate the callable/convertible features in the bonds are obvious directions to pursue. Using this extended modelling framework, an empirical analysis of the model performance would be required to see how well the model-predicted security value changes match the observed changes. Additionally, one could investigate if misrepresentation is a partial explanation of the credit spread puzzle as analyzed in Huang and Huang [29].

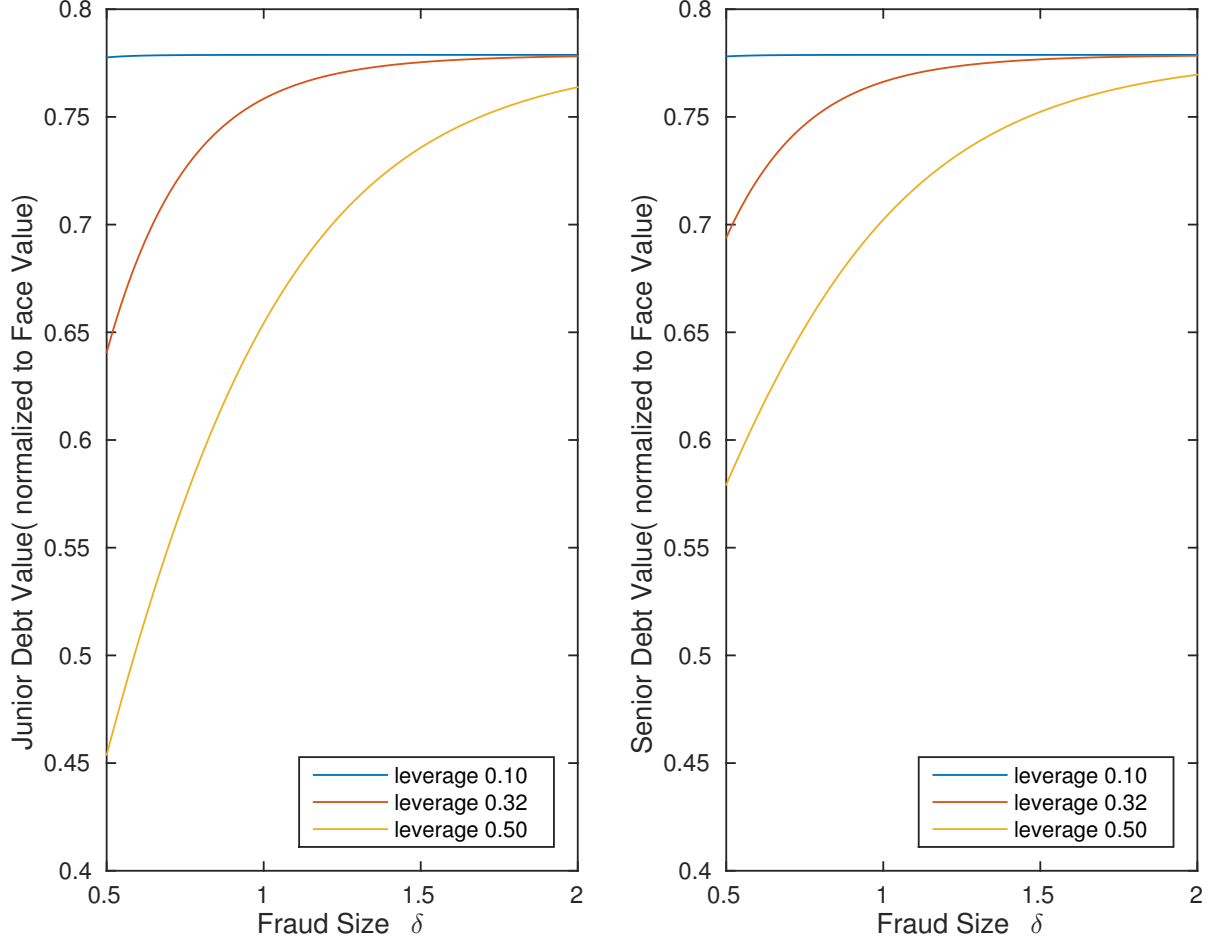


Figure 3.2: Effect of Fraud on Debt Value (FPT) - Fraud size, δ , versus junior/senior zero coupon debt in the left/right panel using the First Passage Time model in Section 3.2. The junior and senior debts have the same maturity date. The quasi-leverage L is defined as quasi market value of debt to the quasi asset value, i.e., $L = (e^{-r(T-t)}K)/(EQ_t + e^{-r(T-t)}K)$. The initial equity value is 100, we use the quasi-leverage ratio to compute the corresponding debt face value K . The face value for junior debt and senior debt are the same, i.e., $K^j = K^s = K/2$. Following the empirical results in Schaefer and Strebulaev[47], we set the leverage ratio L and equity volatility σ^E as 0.10, 0.32, and 0.50 and 25%, 31%, and 42% for AAA, A, and BB credit rating firms, respectively. The rest of the parameter values are $r = 5\%$, $T - t = 5$, $w^j = 82.9\%$, and $w^s = 50.9\%$ (the loss given default w follows [42]).

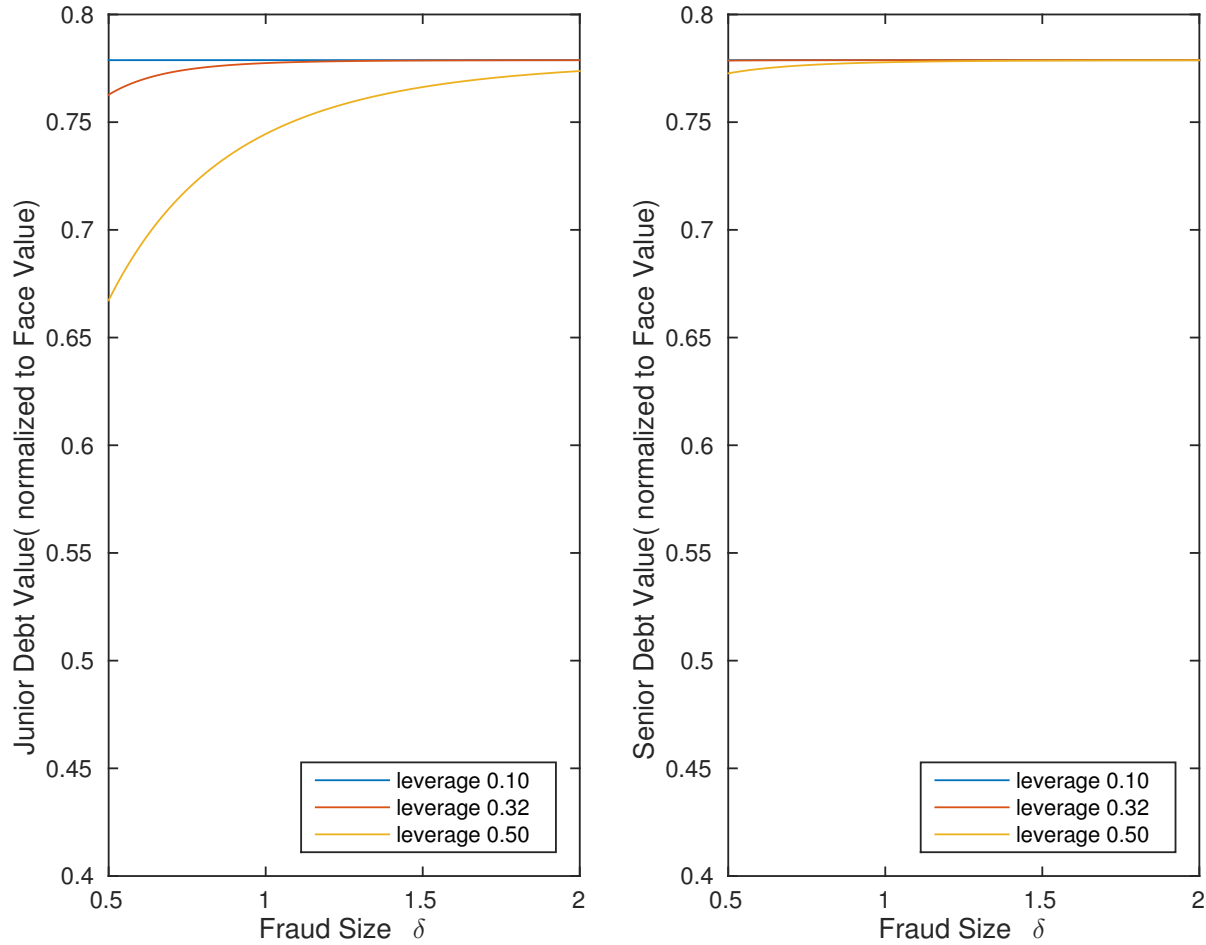


Figure 3.3: Effect of Fraud on Debt Value (SD) - Fraud size, δ , versus junior/senior zero coupon debt in the left/right panel using the Subordinated Debt model. The junior and senior debts have the same maturity date. The quasi-leverage L is defined as quasi market value of debt to the quasi asset value, i.e., $L = (e^{-r(T-t)}F)/(EQ_t + e^{-r(T-t)}F)$. The initial equity value is 100, we use the quasi-leverage ratio to compute the corresponding debt face value F . The face value for junior debt F^j and senior debt F^s are the same, i.e., $F^j = F^s = F/2$. Following the empirical results in Schaefer and Strebulaev[47], we set the leverage ratio L and equity volatility σ^E as 0.10, 0.32, and 0.50 and 25%, 31%, and 42% for AAA, A, and BB credit rating firms, respectively. The rest of the parameter values are $r = 5\%$, and $T - t = 5$.

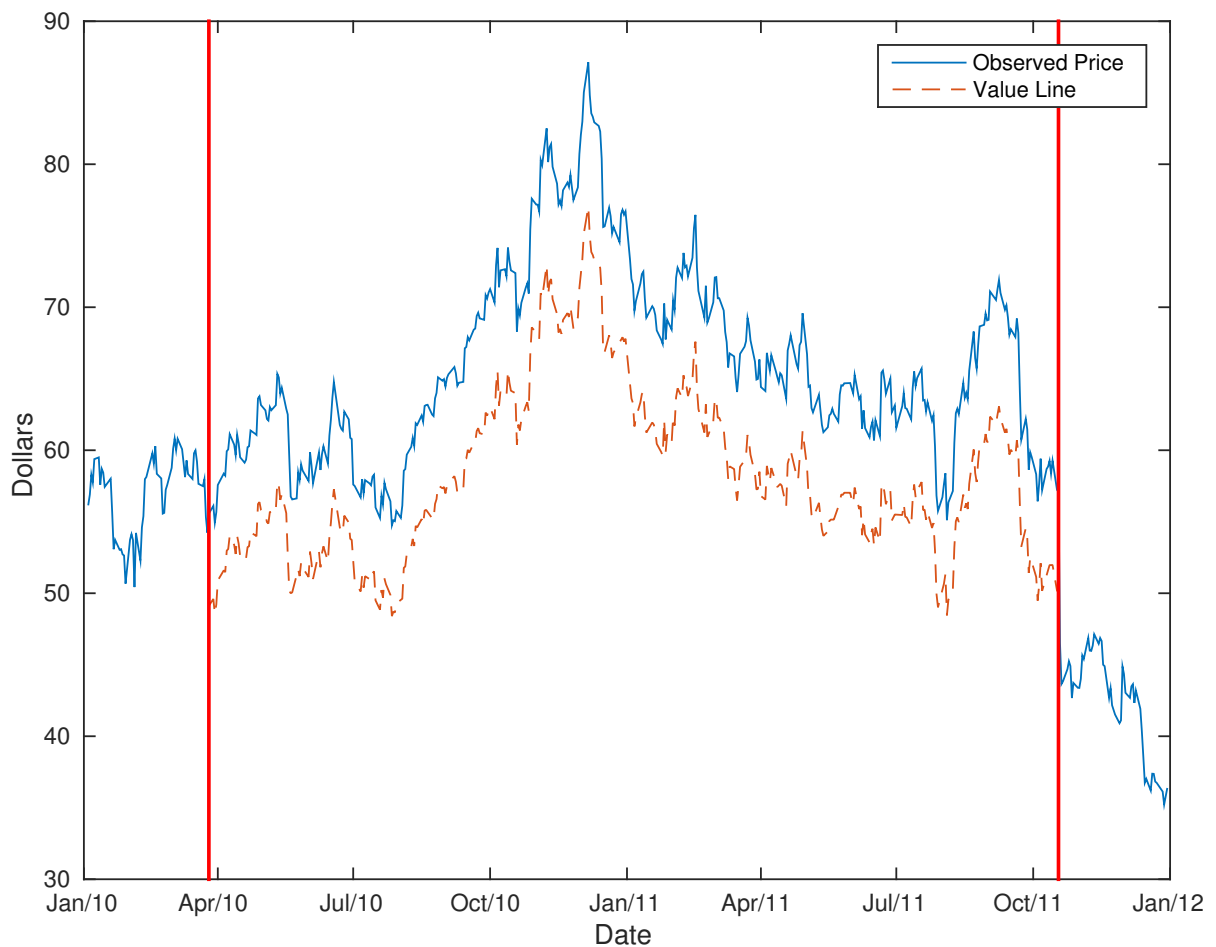


Figure 3.4: Common Shares Value Line - The period between the two vertical lines is the class period. The blue line shows the historical stock price of AEM from January 2010 to December 2011. The value line of stock, which is represented by the red dash line, is constructed by the event study approach (see Section 3.1.1).

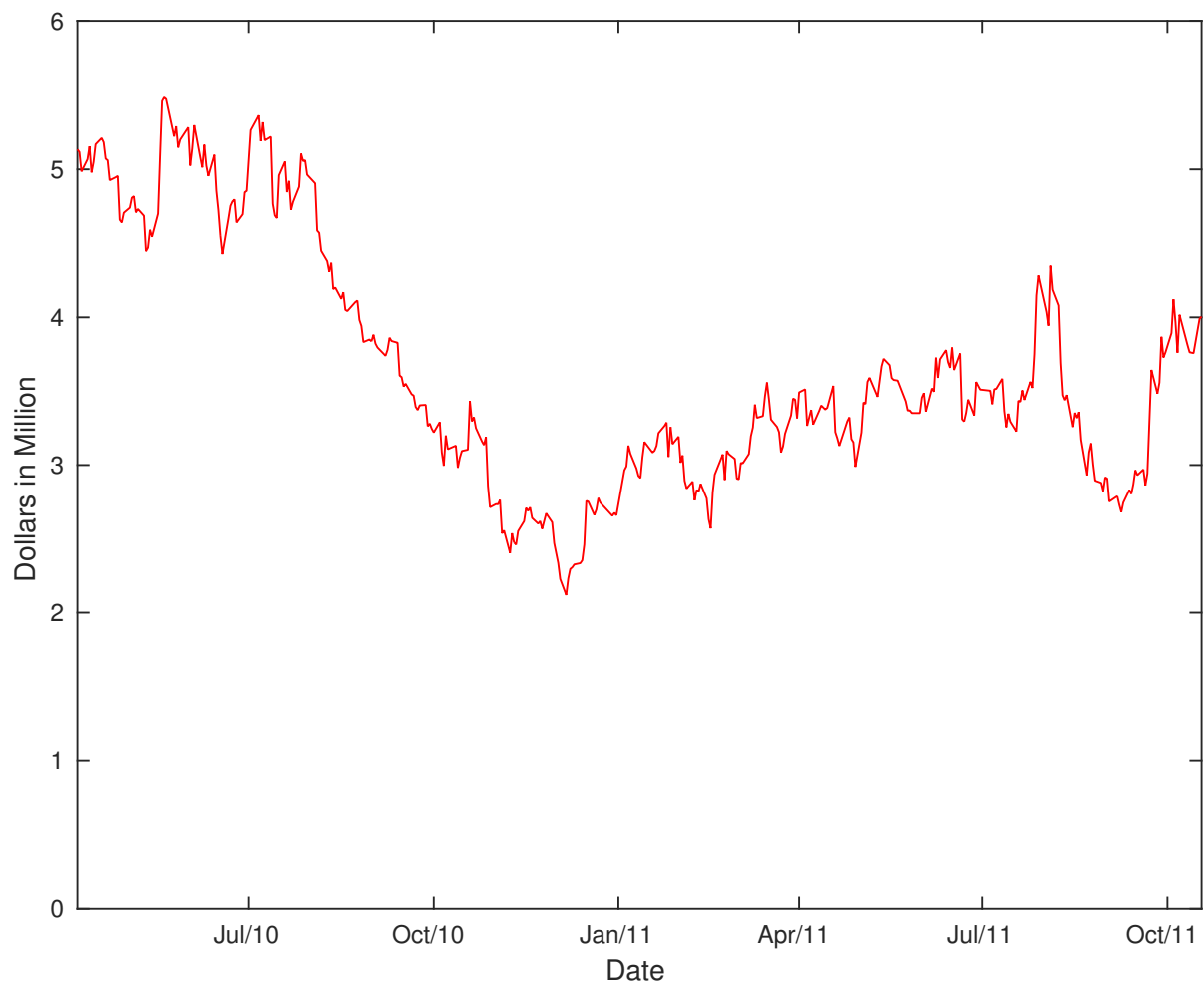


Figure 3.5: Notes Damages Ribbon - The graph shows the damages ribbon of notes during the period from April 7, 2010 to October 18, 2011.

Bibliography

- [1] Agnico-Eagle Mines Limited. Agnico-Eagle Mines Limited, 2010 Annual Report. <http://www.agnicoeagle.com/>, 2011.
- [2] Edward I Altman and Vellore M Kishore. Almost everything you wanted to know about recoveries on defaulted bonds. *Financial Analysts Journal*, pages 57–64, 1996.
- [3] Ray Ball and Philip Brown. An empirical evaluation of accounting income numbers. *Journal of Accounting Research*, 6(2):159–178, 1968.
- [4] Jack Bao, Jun Pan, and Jiang Wang. The illiquidity of corporate bonds. *The Journal of Finance*, 66(3):911–946, 2011.
- [5] Fischer Black and John C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367, 1976.
- [6] Eric Briys and Francois de Varenne. Valuing risky fixed rate debt: An extension. *The Journal of Financial and Quantitative Analysis*, 32(2):239–248, 1997.
- [7] Paul Brockman and H.J. Turtle. A barrier option framework for corporate security valuation. *Journal of Financial Economics*, 67(3):511 – 529, 2003.
- [8] Sudheer Chava, Shunlan Fang, and Saumya Prabhat. Dynamic financial covenant thresholds. In *Working Paper*. 2015.
- [9] Long Chen, David A. Lesmond, and Jason Wei. Corporate yield spreads and bond liquidity. *The Journal of Finance*, 62(1):119–149, 2007.
- [10] Pierre Collin-Dufresne and Robert S. Goldstein. Do credit spreads reflect stationary leverage ratios? *The Journal of Finance*, 56(5):1929–1957, 2001.
- [11] Bradford Cornell and R Gregory Morgan. Using finance theory to measure damages in fraud on the market cases. *UCLA Law Review*, 37:883, 1989.
- [12] Peter Crosbie and Jeff Bohn. Modeling default risk. Research Report, Moody’s KMV Corporation, 2003.
- [13] Michel Crouhy and Dan Galai. The interaction between the financial and investment decisions of the firm: The case of issuing warrants in a levered firm. *Journal of Banking & Finance*, 18(5):861–880, 1994.

- [14] Sergei A Davydenko. When do firms default? a study of the default boundary. Working Paper, 2012.
- [15] Sergei A. Davydenko, Ilya A. Strebulaev, and Xiaofei Zhao. A market-based study of the cost of default. *The Review of Financial Studies*, 25(10):pp. 2959–2999, 2012.
- [16] Jin-Chuan Duan. Maximum likelihood estimation using price data of the derivative contract. *Mathematical Finance*, 4(2):155–167, 1994.
- [17] Jin-Chuan Duan. Correction: Maximum likelihood estimation using price data of the derivative contract (mathematical finance 1994, 4/2, 155167). *Mathematical Finance*, 10(4):461–462, 2000.
- [18] Jin-Chuan Duan, Genevieve Gauthier, and Jean-Guy Simonato. On the equivalence of the KMV and maximum likelihood methods for structural credit risk models. Working Paper, 2005.
- [19] Edward A Dyl. Estimating economic damages in class action securities fraud litigation. *Journal of Forensic Economics*, 12(1):1–11, 1999.
- [20] Edwin J. Elton, Martin J. Gruber, Deepak Agrawal, and Christopher Mann. Explaining the rate spread on corporate bonds. *The Journal of Finance*, 56(1):247–277, 2001.
- [21] Young Ho Eom, Jean Helwege, and Jing-Zhi Huang. Structural models of corporate bond pricing: An empirical analysis. *The Review of Financial Studies*, 17(2):499–544, 2004.
- [22] Jan Ericsson and Joel Reneby. Estimating structural bond pricing models*. *The Journal of Business*, 78(2):707–735, 2005.
- [23] Eugene Fama, Lawrence Fisher, Michael Jensen, and Richard Roll. The adjustment of stock prices to new information. *International Economic Review*, 10, 1969.
- [24] Eugene F. Fama. Efficient capital markets: A review of theory and empirical work*. *The Journal of Finance*, 25(2):383–417, 1970.
- [25] Eugene F. Fama. Efficient capital markets: II. *The Journal of Finance*, 46(5):1575–1617, 1991.
- [26] Jared Tobin Finkelstein. Rule 10b-5 damage computation: Application of financial theory to determine new economic loss. *Fordham Law Review*, 51:838, 1982.
- [27] Daniel R Fischel. Use of modern finance theory in securities fraud cases involving actively traded securities. *The Business Lawyer*, pages 1–20, 1982.
- [28] J Michael Harrison. *Brownian motion and stochastic flow systems*. Wiley New York, 1985.
- [29] Jing-Zhi Huang and Ming Huang. How much of the corporate-treasury yield spread is due to credit risk? *Review of Asset Pricing Studies*, 2(2):153–202, 2012.

- [30] Jing-Zhi Huang and Hao Zhou. Specification analysis of structural credit risk models. AFA 2009 San Francisco Meetings Paper. <http://ssrn.com/abstract=1105640>., 2008.
- [31] E. Philip Jones, Scott P. Mason, and Eric Rosenfeld. Contingent claims analysis of corporate capital structures: an empirical investigation. *The Journal of Finance*, 39(3):611–625, 1984.
- [32] Philip J Leas. The measure of damages in rule 10b-5 cases involving actively traded securities. *Stanford Law Review*, pages 371–398, 1974.
- [33] Hayne E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *The Journal of Finance*, 49(4):1213–1252, 1994.
- [34] Hayne E Leland. Predictions of default probabilities in structural models of debt. *The Credit Market Handbook: Advanced Modeling Issues (H. Gifford Fong, Editor)*, pages 39–64, 2006.
- [35] Hayne E. Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987–1019, 1996.
- [36] Ka Leung Li and Hoi Ying Wong. Structural models of corporate bond pricing with maximum likelihood estimation. *Journal of Empirical Finance*, 15(4):751–777, 2008.
- [37] Francis A. Longstaff and Eduardo S. Schwartz. A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3):789–819, 1995.
- [38] Vassalou Maria and Xing Yuhang. Default risk in equity returns. *The Journal of Finance*, 59(2):831–868.
- [39] Robert C Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.
- [40] Robert C Merton. *Continuous-Time Finance*. Blackwell Boston, 1992.
- [41] Franco Modigliani and Merton H Miller. The cost of capital, corporation finance and the theory of investment. *The American Economic Review*, pages 261–297, 1958.
- [42] Sharon Ou, David Chiu, and Albert Metz. Corporate default and recovery rates, 1920–2010. Moody’s Investors Service, 2011.
- [43] James M. Patell and Mark A. Wolfson. The intraday speed of adjustment of stock prices to earnings and dividend announcements. *Journal of Financial Economics*, 13(2):223–252, 1984.
- [44] Mirela Predescu. Three essays in credit risk. Thesis (Ph.D.)—University of Toronto. 2006.
- [45] Ehud I. Ronn and Avinash K. Verma. Pricing risk-adjusted deposit insurance: An option-based model. *The Journal of Finance*, 41(4):871–896, 1986.

- [46] Mark Rubinstein and Eric Reiner. Breaking down the barriers. *Risk*, 4(8):28–35, 1991.
- [47] Stephen M. Schaefer and Ilya A. Strebulaev. Structural models of credit risk are useful: Evidence from hedge ratios on corporate bonds. *Journal of Financial Economics*, 90(1):1 – 19, 2008.
- [48] David Tabak. Risk disclosures and damages measurement in securities fraud cases. *Securities Reform Act Litigation Reporter*, 2006.
- [49] David Tabak and Frederick Dunbar. Materiality and magnitude: event studies in the courtroom. *Litigation services handbook: the role of the financial expert* New York: John Wiley & Sons, 2001.
- [50] David Tabak and Chudozie Okongwu. Inflation methodologies in securities fraud cases: Theory and practice. NERA Economic Consulting Working Paper, 2002.

Chapter 4

Pricing Warrants with Market Implied Leverage Effect and Dilution Effect

4.1 Introduction

Like an option, a warrant is a derivative that gives its holder the right, but not the obligation, to buy or sell a security at a predetermined price within a specific period. Based on the similarity between a warrant and an option, a straightforward approach to price warrants is to use the Black-Scholes formula¹ [4] and [27]. However, some key differences between warrants and options make pricing warrants more challenging than pricing options. One of the main differences is the dilution effect. While options are usually issued by a third party (e.g., exchange-traded options), warrants are issued by the firm itself. When call warrants are exercised, the firm issues new shares hence increasing the number of the outstanding shares and diluting the ownership stake of existing shareholders. Another difference is the contract life. While exchange-traded options generally have maturities of less than one year, warrants have much longer maturities (e.g., 3-5 years). Moreover, warrants are not as standardized as exchanged-traded options. Some warrants contain some special provisions such as the ability of the warrant issuer to extend warrant maturity (see [25] for pricing warrants with this provision). However, in this chapter we will not consider any special provision when pricing warrants.

Pricing warrants with the dilution effect is first proposed by Galai and Schneller [13]. Similar dilution-adjusted warrant pricing models are discussed in the literatures [11], [21], [8] and [16]. However, other authors [33], [34] and [19] argue that there is no need for dilution adjustment when pricing warrants. Schulz and Trautmann [33] show by simulation that the dilution-adjusted model [13] has relatively small impacts on warrant prices when compared with Black-Scholes formula prices. Their empirical study using German data shows that there is no dilution related bias in pricing warrants. On the other hand, an empirical study using American data by Hauser and Lauterbach [16] shows that the dilution-adjusted models perform significantly better than a model that does not include dilution effect.

In general, the contract life of warrants is much longer than that of exchanged traded options. When pricing options and warrants across the maturity spectrum it is well-known that

¹ Actually, the Black-Scholes formula was originally designed to price warrants (see [2]).

a model with changing volatility is required to accurately match market prices, hence the basic assumption that stock price volatility is constant becomes more questionable as the time horizon of interest increases. The constant elasticity of variance (CEV) model, designed to reflect leverage effects, is proven to be important in warrant pricing. Empirical results [21] and [16] show that CEV model outperforms the constant variance models in warrant pricing. Besides the CEV models, a popular framework in corporate finance to model the leverage effect is through modelling the firm's capital structure (e.g., Merton's framework [28]). In this framework, instead of assuming the stock price follows some stochastic process, the value of the firm is modelled and corporate securities (stocks, bonds, warrants) are valued as contingent claims written on firm value. Crouhy and Galai [9] extended the Galai and Schneller [13] model to incorporate leverage effects in pricing warrants by including zero-coupon debt in the firm's capital structure.

Implementation of the structural framework is challenging because parameter inputs are not directly observable from the market. These parameters include the firm value, firm value volatility, the firm's outstanding debt and its maturity. There is a large literature on the calibration of structural models. A popular approach to calibrating structural models is to utilize the observable stock price and balance-sheet information to infer the firm value and firm value volatility. The latter can be connected to stock price and stock price volatility through a non-linear system as discussed in [31], [20] and [36]. This non-linear system approach has also been studied in warrants pricing literature [35] and [1]. However, recent studies show that the non-linear system method is misspecified [10]. [10] and [12] proposed a more powerful approach², the maximum likelihood estimation (MLE) approach.

Recent empirical studies [18], [7], [5], [6] and [15] show that equity option values contain information on firm leverage effects. Utilizing option price data provides an alternative approach to calibrate structural models as proposed by Hull, Nelken, and White [18]. An advantage of using this approach when compared with the stock price only approach is that it can infer information about a firm's outstanding debt/liability ratio, which can hard to obtain. Moreover, by using this option calibration approach, Geske, Subrahmanyam and Zhou [15] show that structural model (using the compounded option framework [14]) significantly outperforms the Black-Scholes model in pricing options. Particularly, they found that the improvements are greater, the longer the time to expiration of the equity call option. These findings are potentially important for warrant pricing, as warrants are similar to options but with longer maturities.

In this chapter, we introduce a new warrant pricing model to incorporate both dilution and leverage effects by adapting the first passage time structural framework [3], [26] and [36]. As discussed above, much of the warrants pricing literature focusses on one of the dilution or leverage effects, but not both. Our pricing approach is novel as it simultaneously considers both effects. Moreover, by adapting the calibration framework in [18] and [15] we proposed a new calibration method for our warrant pricing model to obtain market implied leverage and dilution effects.

The chapter is organized as follows. The rest of Section 4.1 reviews the First Passage Time structural (FPT) framework. Section 4.2 gives the option price under the FPT framework.

²Studies [12] and [24] show that that the MLE approach outperforms the non-linear system approach in corporate bond pricing.

Section 4.3 discusses the calibration of FPT framework using market implied volatility skew. Section 4.4 extended the FPT structural framework to include warrants in the firm's capital structure. Section 4.5 gives the call option price under the warrant extended structural framework. It proves a connection between option price and warrant price and hence the calibration method for the warrant pricing model. Section 4.6 shows the implementation and model performance of the pricing models. A conclusion is given in Section 4.7.

4.1.1 The First Passage Time Structural Model

In this section we review the first passage time model in [36]. Under the first passage time model, it is assumed that default is exogenous and that there is a threshold value K ; when the firm value V^F reaches the constant boundary K , the firm enters financial distress, and simultaneously defaults on all of its obligations. The firm value V^F is assumed to follow dynamic

$$dV_t^F = rV_t^F dt + \sigma^F V_t^F dW_t, \quad (4.1)$$

where r is the risk-free rate, σ^F is the firm volatility and W_t is a standard Brownian motion under the risk-neutral measure. Assume that the Modigliani-Miller Theorem [30] holds: the firm value is independent of the capital structure of the firm. This paper focuses on warrant/option pricing under this structural framework. Please refer to C.1 for a brief discussion of this structural framework with bankruptcy cost.

Let \bar{T} be the average maturity date of debts weighted by the face value of debts. If the firm defaults before \bar{T} , the equity value is assumed to be wiped out. Following the Black and Cox [3] framework, we evaluate the firm's equity value as a European down-and-out call option³ written on firm value V^F with strike price and knock out barrier equal to K . Let $L \equiv K/V_t^F$ be the measure of leverage. The equity value EQ_t is given by

$$EQ_t = V_t^F (\Phi(d_1) - Le^{-r(\bar{T}-t)}\Phi(d_2) - L^{2\lambda}\Phi(d_3) + e^{-r(\bar{T}-t)}L^{2\lambda-1}\Phi(d_4)), \quad (4.2)$$

where

$$d_1 = (-\log(L) + (r + \sigma^{F^2}/2)(\bar{T} - t))/(\sigma^F \sqrt{\bar{T} - t}), \quad (4.3)$$

$$d_2 = d_1 - \sigma^F \sqrt{\bar{T} - t}, \quad (4.4)$$

$$d_3 = (\log(L) + (r + \sigma^{F^2}/2)(\bar{T} - t))/(\sigma^F \sqrt{\bar{T} - t}), \quad (4.5)$$

$$d_4 = d_3 - \sigma^F \sqrt{\bar{T} - t}, \quad (4.6)$$

and

$$\lambda = (r + \sigma^{F^2}/2)/\sigma^{F^2}. \quad (4.7)$$

4.2 Option Price under the First Passage Time Framework

The European call option price under the first passage time framework is derived in this section. Let C_t be the time- t value of a call option with strike price K^O and maturity date T_O . As in [9],

³Research on barrier options valuation can be found in [32] and [29].

we only consider the case where $T_O \leq \bar{T}$. Under this framework, share price is a down and out call option of the firm value V_t^F , i.e., $S(V_t^F, t, \bar{T}) = EQ(V_t^F, t, \bar{T})/N$, where N is the number of outstanding shares. The exercise criteria of the call option can be defined using $V_{T_O}^F$, the time- T_O firm value. Let \bar{V}_O^F be the threshold value such that, $S(\bar{V}_O^F, t, \bar{T}) = K^O$. Since $EQ(V_t^F, t, \bar{T})$ is a monotonic function of V_t^F , the call option is in the money when $V_{T_O}^F \geq \bar{V}_O^F$ and out of the money otherwise. Moreover, the option being exercised at T_O implies that no default event happened before T_O . Hence, the exercise criteria of the call option is $V_{T_O}^F \geq \bar{V}_O^F$ and $m_{T_O}^{V^F} > K$, where $m_{T_O}^{V^F} = \min\{V_s^F, t < s < T_O\}$.

We derive the call option value using the martingale approach, i.e., compute the expected discounted pay-off of the call option under risk neutral measure. It is easy to show that a solution to the SDE given in equation (4.1) is

$$V_s^F = V_t^F \exp\left((r - \frac{1}{2}\sigma^{F^2})(s - t) + \sigma^F(W_s - W_t)\right), \quad (4.8)$$

for $t \leq s$. Let $\hat{W}_{s-t} \equiv (r - \frac{1}{2}\sigma^{F^2})(s - t) + \sigma^F(W_s - W_t)$. \hat{W}_v is a $(r - 0.5\sigma^{F^2}, \sigma^F)$ Brownian motion starting at zero. Equation (4.8) becomes

$$V_s^F = V_t^F e^{\hat{W}_{s-t}}. \quad (4.9)$$

Let $\bar{W}_O \equiv \log(\bar{V}_O^F/V_t^F)$ and $W^K \equiv \log(K/V_t^F) = \log(L)$. Recall that the firm defaults when the firm value V_t^F reaches the default boundary K . Let $\hat{m}_u \equiv \inf\{\hat{W}_v, 0 \leq v \leq u\}$. $\hat{m}_{T_O-t} \leq W^K$ implies the firm defaults before T_O . Given that the firm does not default before T_O , i.e., $m_{T_O}^{V^F} > K$, the call option would be exercised if $\hat{W}_{T_O-t} \geq \bar{W}_O$.

Summarizing the analysis above, the time- t call option price, C_t , is

$$\begin{aligned} C_t &= e^{-r(T_O-t)} EQ\left[\left(\frac{EQ(V_{T_O}^F, T_O, \bar{T})}{N} - K^O\right) I_{\{V_{T_O}^F \geq \bar{V}_O^F, m_{T_O}^{V^F} > K\}}\right] \\ &= e^{-r(T_O-t)} \int_{\bar{W}_O}^{+\infty} \left(\frac{EQ(V_t^F e^x, T_O, \bar{T})}{N} - K^O\right) g_{T_O-t}(x, W^K) dx, \end{aligned} \quad (4.10)$$

where g_{T_O-t} is a conditional distribution function given by equation (C.10).

4.3 Calibration of the Model Using Implied Volatility

The challenges in using the option price in equation (4.10) is that some of its parameters are not directly observable from the market, such as firm value V_t^F , firm volatility σ^F and the default boundary K . Moreover, it would be difficult to calculate the default boundary K for firms with complicated capital structure. In order to solve this issue, we derive a calibration method following Hull, Nelken and White's work [18]. The solution is to use the information contained in option prices to infer the non-observable firm value V_t^F , firm volatility σ^F as well as the default boundary K .

To simplify things, we introduce two ratios α and κ , which represent the moneyness of the option with respect to firm value and share value, respectively. Let $\bar{V}_O^F = \alpha V_t^F$ and $K^O =$

$\kappa EQ_t/N$. A relationship between α and κ will be derived in this section. Given these notations, $\bar{W}_O = \log \alpha$ and the time- t call option value, C_t , can be written as

$$C_t = e^{-r(T_O-t)} \frac{V_t^F}{N} \int_{\log \alpha}^{+\infty} (e^x \Phi(\tilde{d}_1) - e^{-r(\bar{T}-T_O)} L \Phi(\tilde{d}_2) - e^{x(1-2\lambda)} L^{2\lambda} \Phi(\tilde{d}_3) + e^{-r(\bar{T}-T_O)-x(2\lambda-2)} L^{2\lambda-1} \Phi(\tilde{d}_4)) g_{T_O-t}(x, \log L) dx - e^{-r(T_O-t)} \kappa \frac{EQ_t}{N} G_{T_O-t}(\log \alpha, \log L), \quad (4.11)$$

where

$$\tilde{d}_1 = (-\log(L) + x + (r + \sigma^{F^2}/2)(\bar{T} - T_O)) / (\sigma^F \sqrt{\bar{T} - T_O}), \quad (4.12)$$

$$\tilde{d}_2 = \tilde{d}_1 - \sigma^F \sqrt{\bar{T} - T_O}, \quad (4.13)$$

$$\tilde{d}_3 = (\log(L) - x + (r + \sigma^{F^2}/2)(\bar{T} - T_O)) / (\sigma^F \sqrt{\bar{T} - T_O}), \quad (4.14)$$

$$\tilde{d}_4 = \tilde{d}_3 - \sigma^F \sqrt{\bar{T} - T_O}, \quad (4.15)$$

and $G_{T_O-t}(x, y)$ is the joint distribution function given by equation (C.9). Notice that under the Merton framework, the call option value only depends on the firm value distribution at two given points in time, maturity time T_O and \bar{T} . However, this does not hold under the first passage time framework. Hence, the call option expression cannot be simplified to one involving the bivariate normal distribution as in Geske [14].

Let v be the Black-Scholes implied volatility of the call option, i.e., v satisfies equation

$$C_t = \frac{EQ_t}{N} \Phi(l_1) - \kappa \frac{EQ_t}{N} e^{-r(T_O-t)} \Phi(l_2), \quad (4.16)$$

where

$$l_1 = (-\log(\kappa) + (r + v^2/2)(T_O - t)) / (v \sqrt{T_O - t}), \quad (4.17)$$

$$l_2 = l_1 - v \sqrt{T_O - t}. \quad (4.18)$$

Equating the right hand side of equation (4.11) and equation (4.16), and substituting for EQ_t as given in equation (4.2), we get

$$\begin{aligned} & e^{-r(T_O-t)} \int_{\log \alpha}^{+\infty} (e^x \Phi(\tilde{d}_1) - e^{-r(\bar{T}-T_O)} L \Phi(\tilde{d}_2) - e^{x(1-2\lambda)} L^{2\lambda} \Phi(\tilde{d}_3) \\ & + e^{-r(\bar{T}-T_O)-x(2\lambda-2)} L^{2\lambda-1} \Phi(\tilde{d}_4)) g_{T_O-t}(x, \log L) dx \\ & - e^{-r(T_O-t)} \kappa G_{T_O-t}(\log \alpha, \log L) (\Phi(d_1) - L e^{-r(\bar{T}-t)} \Phi(d_2) - L^{2\lambda} \Phi(d_3) + e^{-r(\bar{T}-t)} L^{2\lambda-1} \Phi(d_4)) \\ & = (\Phi(l_1) - \kappa e^{-r(T_O-t)} \Phi(l_2)) (\Phi(d_1) - L e^{-r(\bar{T}-t)} \Phi(d_2) - L^{2\lambda} \Phi(d_3) + e^{-r(\bar{T}-t)} L^{2\lambda-1} \Phi(d_4)). \end{aligned} \quad (4.19)$$

By the definition of \bar{V}_O^F , we have

$$\kappa \frac{EQ_t}{N} = \frac{\bar{V}_O^F}{N} (\Phi(d_{1,T_O}) - e^{-r(\bar{T}-T_O)} \frac{L}{\alpha} \Phi(d_{2,T_O}) - \frac{L^{2\lambda}}{\alpha^{2\lambda}} \Phi(d_{3,T_O}) + e^{-r(\bar{T}-T_O)} \frac{L^{2\lambda-1}}{\alpha^{2\lambda-1}} \Phi(d_{4,T_O})), \quad (4.20)$$

where

$$d_{1,T_0} = (-\log(L) + \log(\alpha) + (r + \sigma^F/2)(\bar{T} - T_0))/(\sigma^F \sqrt{\bar{T} - T_0}), \quad (4.21)$$

$$d_{2,T_0} = d_{1,T_0} - \sigma^F \sqrt{\bar{T} - T_0}, \quad (4.22)$$

$$d_{3,T_0} = (\log(L) - \log(\alpha) + (r + \sigma^F/2)(\bar{T} - T_0))/(\sigma^F \sqrt{\bar{T} - T_0}), \quad (4.23)$$

$$d_{4,T_0} = d_{3,T_0} - \sigma^F \sqrt{\bar{T} - T_0}. \quad (4.24)$$

Given the expression of EQ_t in equation (4.2), the relation between κ and α can be derived, i.e.,

$$\kappa = \alpha \frac{\Phi(d_{1,T_0}) - e^{-r(\bar{T}-T_0)} \frac{L}{\alpha} \Phi(d_{2,T_0}) - \frac{L^{2\lambda}}{\alpha^{2\lambda}} \Phi(d_{3,T_0}) + e^{-r(\bar{T}-T_0)} \frac{L^{2\lambda-1}}{\alpha^{2\lambda-1}} \Phi(d_{4,T_0})}{\Phi(d_1) - L e^{-r(\bar{T}-t)} \Phi(d_2) - L^{2\lambda} \Phi(d_3) + e^{-r(\bar{T}-t)} L^{2\lambda-1} \Phi(d_4)}. \quad (4.25)$$

With a set of parameters, i.e., L , σ^F , r , \bar{T} and T_0 , we can get the relationship between κ and the Black-Scholes implied volatility ν using equations (4.19) and (4.25). This connection leads to a model implied volatility skew (see Figure 4.1). This plot shows that a curve of the model implied volatility skew can be identified by using two points, i.e., given two data points on the surface of κ versus ν , there is a unique model implied volatility skew that passes through these two points. The model implied volatility skew under the Merton's framework is given by Figure 4.2. As pointed out by Hull, Nelken and White, under reasonable values for model parameters the Merton framework fails to generate model implied volatility skews that are consistent with observations; the model implied volatility skew is too flat when compared with the one observed from the market. However, as shown in Figure 4.1, the first passage framework can generate a much larger range of volatility skew using reasonable parameters.

With equations (4.19) and (4.25), we can calibrate our structural model using two Black-Scholes implied volatilities ν_1 and ν_2 .⁴ For a given \bar{T} , a relationship between the leverage L and firm volatility σ^F can be derived using ν_1 . Given the second implied volatility ν_2 , we can solve for both L and σ^F . Using the observed equity value EQ_t , the firm value V_t^F can be calculated by solving equation (4.2).

4.4 Debt, Warrants, and Common Shares Capital Structure

In this section, we extend the first-passage-time capital structure framework to include warrants. Consider a firm financed with debt, equity and warrants. When default happens, the equity holders get wiped out, and hence the warrants become worthless. The number of outstanding warrants is M and each warrant promises the holder the right to purchase one common share at strike price K^W on maturity date T_W . Let X_t be the time- t value of a warrant. For simplicity, the warrant is assumed to be European style. We only consider the case $T_W \leq \bar{T}$ as in [9]. Moreover, this paper only considers the case of block exercise as in [13] and [9], i.e., the warrant holders either exercise all warrants or no exercises happens. A benchmark firm, whose

⁴ ν_1 and ν_2 are implied volatilities derived from two different options. For example, they can be the implied volatilities of two options with same maturities but different strikes.

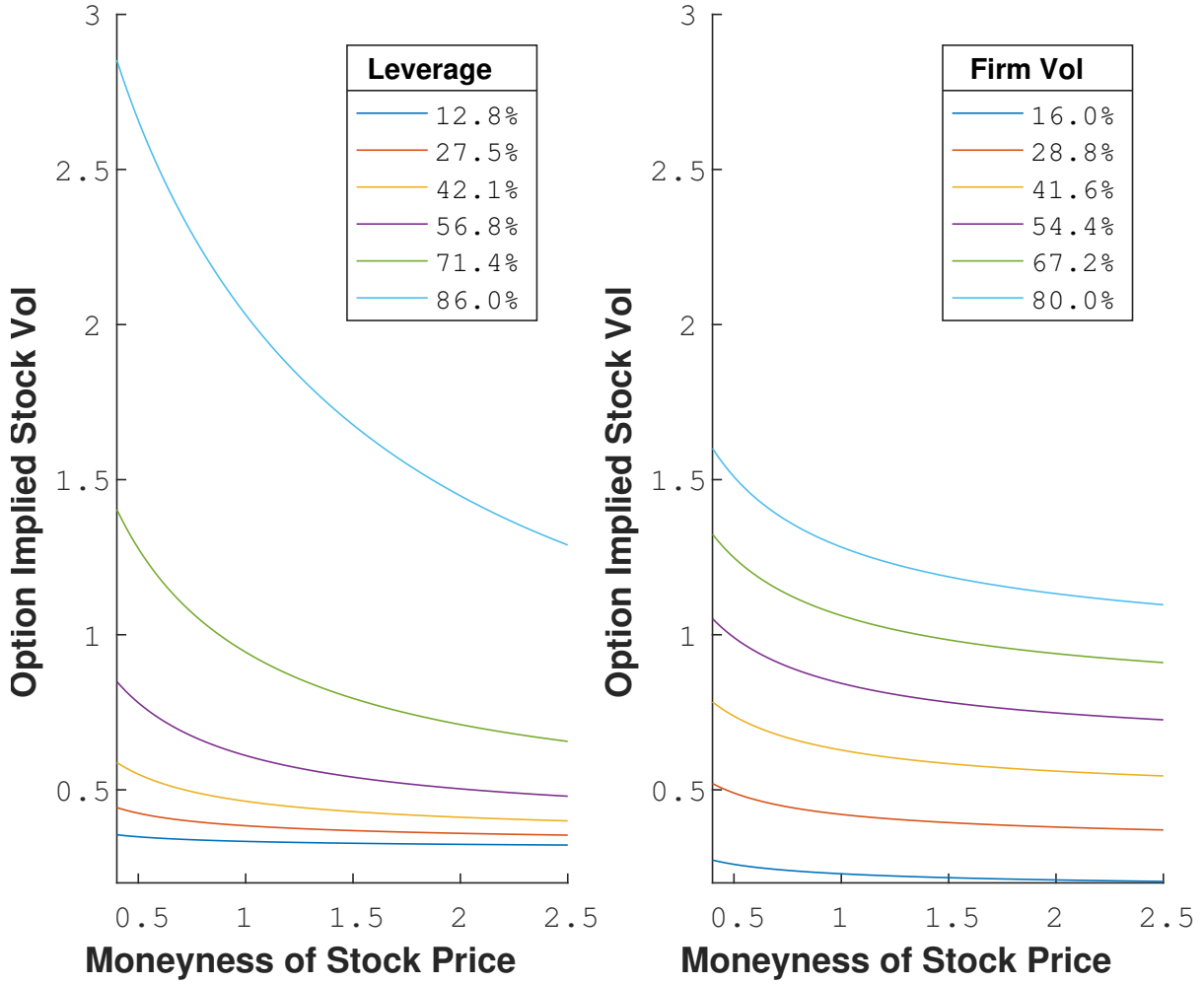


Figure 4.1: First-passage-time model implied volatility skew - Moneyiness of stock price κ versus implied stock volatility ν . Parameters input are below: $r = 5\%$, $t = 0$, $T_O = 0.33$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

value is denoted by V^{F*} , is an identical firm financed entirely by common shares, i.e., the value of the two firms are the same except on the date when warrants are exercised. So $V_t^F = V_t^{F*}$ for $t \leq T_W$. The firm value for $t \in (T_W, \bar{T})$ depends on whether the warrants are exercised, i.e.,

$$V_t^F = \begin{cases} (1 + MK^W/V_{T_W}^{F*})V_t^{F*} & \text{if warrants exercised} \\ V_t^{F*} & \text{otherwise} \end{cases}.$$

The cash from exercising the M warrants are assumed to be reinvested in the firm, increasing its size proportionally. After the maturity date T_W , the capital structure of the firm consists only of debt, equity and bankruptcy cost as described in Section 4.1.1, no matter whether the warrants are exercised or not. Let $D(V_t^F, t, \bar{T})$ and $EQ(V_t^F, t, \bar{T})$ denote the time- t debt value and equity value, respectively. The pricing formula for $EQ(V_t^F, t, \bar{T})$ under the first passage time framework is given by equations (4.2). In the case where a firm has only one zero-coupon

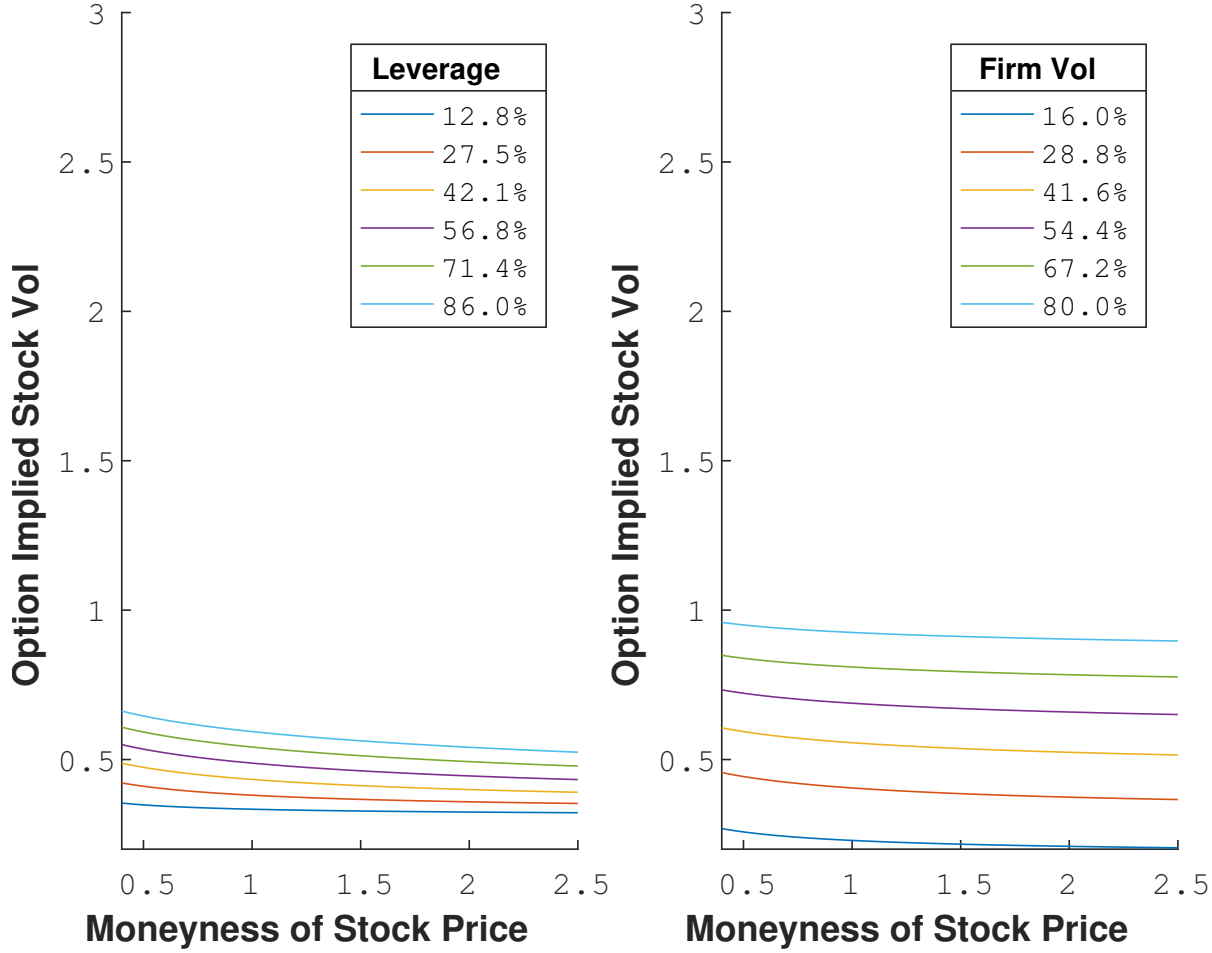


Figure 4.2: Merton model implied volatility skew - Moneyness of stock price κ versus implied stock volatility ν . Parameters input are below: $r = 5\%$, $t = 0$, $T_O = 0.33$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

bond outstanding, the debt value is just the value of the zero-coupon bond, whose price is given by equation (C.2). Given the number of outstanding shares, N , the time- t value of stock is $S(V_t^F, t, \bar{T}) = EQ(V_t^F, t, \bar{T})/N$.

Since exercising of the warrants will change the firm value V^F . This will also affect value of debt and common shares, whose values are functions of V^F . Hence, for this capital structure the exercise criteria for warrants will be defined based on the firm value instead of the firm's stock price, the exercise criteria for a pure equity firm. In other words, even if $S_{T_W} > K^W$ on maturity date T_W , it may not be optimal to exercise the warrants because the cash from warrant exercise will partially flow to the debt value and hence it is possible to cause the stock price to drop below the strike price right after warrant exercise. To show this fact we examine the stock price immediately after warrant exercise T_W^+ . Assume that there is no default before T_W , i.e., the firm value does not reach the default boundary K before T_W . Let $S_{T_W^+}^W$ be the stock price right after warrant exercise and $S_{T_W^+}^{NW}$ be the stock price without exercise. Then the time- T_W

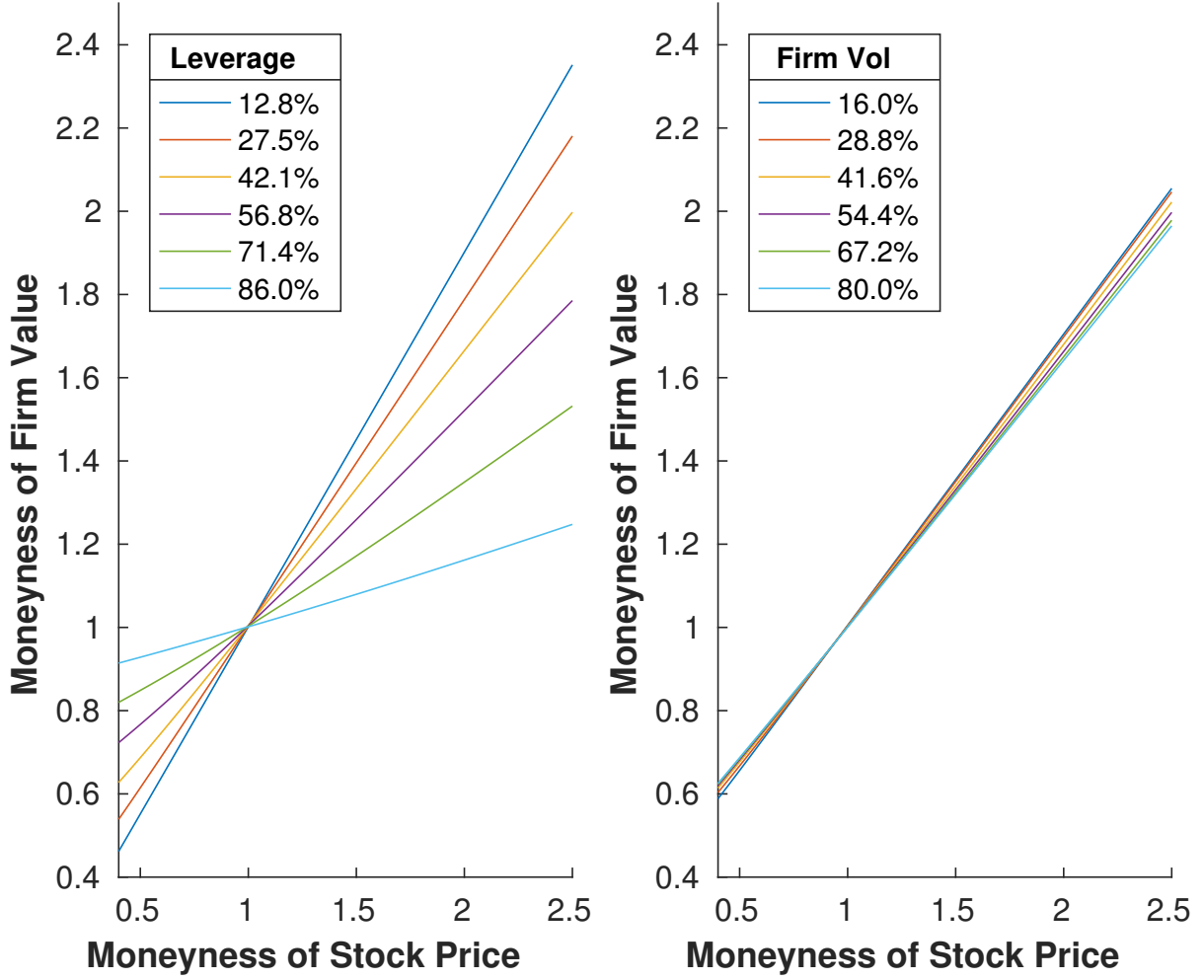


Figure 4.3: First-passage-time model - Moneyness of stock price κ versus Moneyness of firm value α . Parameters input are below: $r = 5\%$, $t = 0$, $T_O = 0.33$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

value of common shares is given by

$$S_{T_w^+} = \begin{cases} \frac{EQ(V_{T_w}^{F*} + MK^W, T_w, \bar{T})}{N + M} \equiv S_{T_w^+}^W & \text{if warrants exercised} \\ \frac{EQ(V_{T_w}^{F*}, T_w, \bar{T})}{N} \equiv S_{T_w^+}^{NW} & \text{if warrants not exercised} \end{cases}.$$

To determine the warrant exercise criteria, we need to find the threshold value of the firm, \bar{V}^F , such that $S_{T_w^+}^W(\bar{V}^F, T_w, \bar{T}) = K^W$. If $V_{T_w}^F \geq \bar{V}^F$ the warrant holders should exercise their warrants, otherwise exercising the warrants would generate a negative payoff.

When default happens, the equity value becomes zero and hence the time- T_w value of equity is zero. Summarizing the discussion above, Let $m_{T_w}^{V^{F*}} = \min\{V_s^{F*}, t < s < T_w\}$. Let \bar{w} be

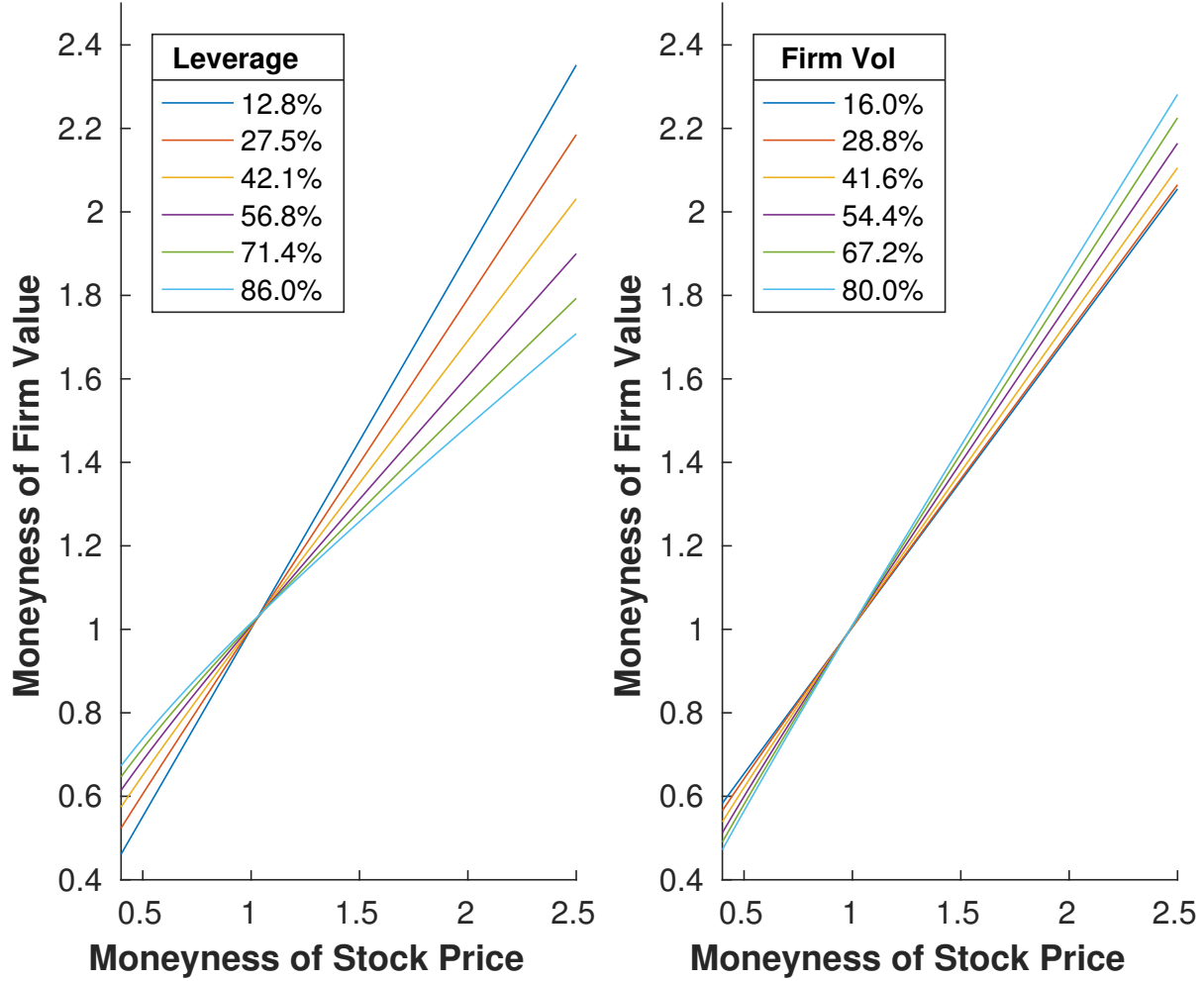


Figure 4.4: Merton model - Moneyness of stock price κ versus Moneyness of firm value α . Parameters input are below: $r = 5\%$, $t = 0$, $T_O = 0.33$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are choosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^*e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^*e^{r(\bar{T}-t)}$).

the loss given default of all the debts of the firm. Table 4.1 gives the time- T_W equity value and debt value, which is determined by the sample path of firm value V^F .

Based on Table 4.1, we can calculate the value of debt and equity at time $t < T_W$ by the expected discounted value of the time- T_W payoff under the risk-neutral measure. The benchmark firm value V_t^{F*} dynamics are

$$dV_t^{F*} = rV_t^{F*}dt + \sigma^F V_t^{F*}dW_t, \quad (4.26)$$

where r is the risk-free rate, assumed to be constant, and W_t is a standard Brownian motion under the risk neutral measure. This SDE has an analytic solution

$$V_s^{F*} = V_t^{F*} \exp\left(\left(r - \frac{1}{2}\sigma^{F^2}\right)(s-t) + \sigma^F(W_s - W_t)\right), \quad (4.27)$$

for $t \leq s$. Let $\hat{W}_{s-t} \equiv \left(r - \frac{1}{2}\sigma^{F^2}\right)(s-t) + \sigma^F(W_s - W_t)$. \hat{W}_v is a $(r - 0.5\sigma^{F^2}, \sigma^F)$ Brownian motion

Table 4.1: Value of debt and equity at T_W^+ .

	No Default before T_W	Default before T_W
	Warrants exercised $V_{T_W}^{F*} \geq \bar{V}^F \ \& \ m_{T_W}^{V^{F*}} > K$	Warrants not exercised $V_{T_W}^{F*} < \bar{V}^F \ \& \ m_{T_W}^{V^{F*}} > K$
	$m_{T_W}^{V^{F*}} \leq K$	
X_{T_W}	$S_{T_W}^W - K^W$	0
S_{T_W}	$EQ(V_{T_W}^{F*} + MK^W, T_W, \bar{T})/(N + M)$	$EQ(V_{T_W}^{F*}, T_W, \bar{T})/N$
D_{T_W}	$D(V_{T_W}^{F*} + MK^W, T_W, \bar{T})$	$D(V_{T_W}^{F*}, T_W, \bar{T})$
		$\bar{w}K e^{-r(\bar{T}-T_W)}$

starting at zero. And equation (4.27) become

$$V_s^{F*} = V_t^{F*} e^{\hat{W}_{s-t}}, \quad (4.28)$$

Let $\bar{W} \equiv \log(\bar{V}^F/V_t^{F*})$ and $W^K \equiv \log(K/V_t^{F*})$. Reminder that the firm defaults when the firm value V_t^F reach the default boundary K . Let $m_t \equiv \inf\{\hat{W}_s, s \leq t\}$. $m_{T_W-t} \leq W^K$ implies the firm defaults before T_W . Given that the firm does not default before T_W , if $\hat{W}_{T_W-t} \geq \bar{W}$, warrants should be exercised, otherwise warrants should not be exercised.

Summarizing the results above, we can compute the stock price and debt value at time t . The time- t warrant price is

$$\begin{aligned} X_t &= e^{-r(T_W-t)} EQ \left[\left(\frac{EQ(V_{T_W}^{F*} + MK^W, T_W, \bar{T})}{N + M} - K^W \right) I_{\{V_{T_W}^{F*} \geq \bar{V}^F, m_{T_W}^{V^{F*}} > K\}} \right] \\ &= e^{-r(T_W-t)} \int_{\bar{W}}^{\infty} \left(\frac{EQ(V_t^{F*} e^x + MK^W, T_W, \bar{T})}{N + M} - K^W \right) g_{T_W-t}(x, W^K) dx \end{aligned} \quad (4.29)$$

The time- t stock value is

$$\begin{aligned} S_t &= e^{-r(T_W-t)} EQ \left[\left(\frac{EQ(V_{T_W}^{F*} + MK^W, T_W, \bar{T})}{N + M} \right) I_{\{V_{T_W}^{F*} \geq \bar{V}^F, m_{T_W}^{V^{F*}} > K\}} \right. \\ &\quad \left. + \left(\frac{EQ(V_{T_W}^{F*}, T_W, \bar{T})}{N} \right) I_{\{V_{T_W}^{F*} < \bar{V}^F, m_{T_W}^{V^{F*}} > K\}} \right] \\ &= e^{-r(T_W-t)} \left(\int_{\bar{W}}^{\infty} \frac{EQ(V_t^{F*} e^x + MK^W, T_W, \bar{T})}{N + M} g_{T_W-t}(x, W^K) dx \right. \\ &\quad \left. + \int_{W^K}^{\bar{W}} \frac{EQ(V_t^{F*} e^x, T_W, \bar{T})}{N} g_{T_W-t}(x, W^K) dx \right). \end{aligned} \quad (4.30)$$

And the time- t debt value is

$$\begin{aligned} D_t &= e^{-r(T_W-t)} EQ [D(V_{T_W}^{F*} + MK^K, T_W, \bar{T}) I_{\{V_{T_W}^{F*} \geq \bar{V}^F, m_{T_W}^{V^{F*}} > K\}} + D(V_{T_W}^{F*}, T_W, \bar{T}) I_{\{V_{T_W}^{F*} < \bar{V}^F, m_{T_W}^{V^{F*}} > K\}} \\ &\quad + \bar{w}K e^{-r(\bar{T}-T_W)} I_{\{m_{T_W}^{V^{F*}} \leq K\}}] \\ &= e^{-r(T_W-t)} \left(\int_{\bar{W}}^{\infty} D(V_t^{F*} e^x + MK^K, T_W, \bar{T}) g_{T_W-t}(x, W^K) dx \right. \\ &\quad \left. + \int_{W^K}^{\bar{W}} D(V_t^{F*} e^x, T_W, \bar{T}) g_{T_W-t}(x, W^K) dx + \bar{w}K e^{-r(\bar{T}-T_W)} Q\{m_{T_W-t} \leq W^K\} \right), \end{aligned} \quad (4.31)$$

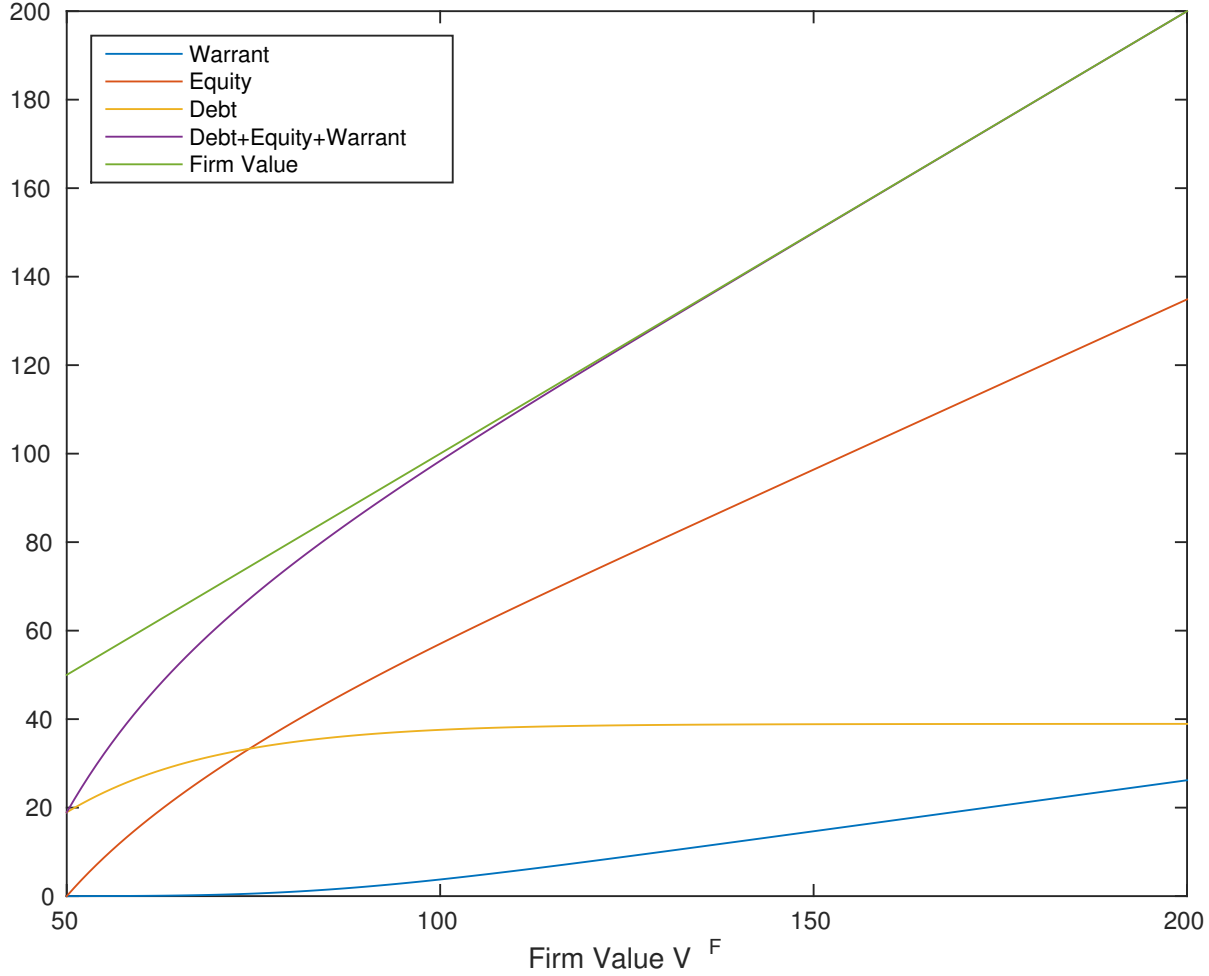


Figure 4.5: Firm Value and Capital Structures - Firm value V_t^F versus warrant value X_t , equity value EQ_t , debt value D_t and the sum of X_t , EQ_t and D_t . The bankruptcy costs BC_t are the difference between V_t^F and $D_t + EQ_t + X_t$. Parameters input are below: $\sigma^F = 21\%$, $r = 5\%$, $t = 0$, $T_W = 1$, $T = 5$, $K = 50$, $K^W = 5$, $N = 10$, $M = 3$ and $w = 48.67\%$ (parameter inputs r and w are chosen following Huang and Huang[17]).

where $Q\{m_t \leq m\}$ is given by equation (C.12).

Since firm value is the sum of the values of debt, equity and the expected bankruptcy costs, the time- t value of the expected bankruptcy cost is

$$BC_t = V_t^{F*} - MX_t - NS_t - D_t, \quad (4.32)$$

where X_t , S_t and D_t are computed using Equations (4.29), (4.30) and (4.31), respectively. A numerical example is give by Figure 4.5. The numerical results shows that the expected bankruptcy cost BC_t defined in 4.32 is a decreasing, convex function of the firm value V_t^{F*} . It has the same properities as the bankruptcy cost defined in [22] and [23], i.e., BC_t satisfies the

boundary conditions

$$\text{at } V_t^{F*} = K, BC_t = K - e^{-r(T-t)}(1-w)F, \text{ and} \quad (4.33)$$

$$\text{as } V_t^{F*} \rightarrow \infty, BC_t \rightarrow 0. \quad (4.34)$$

It is easy to see that the boundary condition (4.33) holds. Boundary condition (4.34) also holds: when the firm value V_t^{F*} becomes very large, the debt value D_t approaches to $e^{-r(\bar{T}-t)}F$, which is the value of risk-free debt with the same maturity and face value, and the sum of equity value EQ_t and warrants value approaches to $V_t^{F*} - Ke^{-r(T-t)}$, which is the upper bound for a European call option price under the Black Scholes's framework, and hence the present value of the bankruptcy costs BC_t approaches to zero, because $K = F$.

4.5 Calibration of the Warrants Pricing Model

As an extension of the discussion in Section 4.3, we discuss the calibration of our warrant pricing model in this section. Results of both Merton and the First Passage Time frameworks are shown. However, the calibration approach discussed in this section should be straightforward to implement in other structural frameworks.

4.5.1 Connecting Firm Value and Equity Value by Solving a Non-Linear System

For a given default boundary K , the connection between the firm value and equity value could be summarized by the following non-linear system.

$$S_t = e^{-r(T_W-t)} \left(\int_{\bar{W}}^{\infty} \frac{EQ(V_t^{F*}e^x + MK^W, T_W, \bar{T})}{N+M} g_{T_W-t}(x, W^K) dx + \int_{W^K}^{\bar{W}} \frac{EQ(V_t^{F*}e^x, T_W, \bar{T})}{N} g_{T_W-t}(x, W^K) dx \right). \quad (4.35)$$

$$\sigma^S = \sigma^F \frac{\partial S_t}{\partial V_t^F} \frac{V_t^F}{S_t} \quad (4.36)$$

In the above formula, it is assumed that the stock volatility σ^S is known or given.

Another approach to calibrate the model using the above non-linear system is the MLE method (see Duan[10], Ericsson and Reneby[12], and Zhou and Reesor[36] for details). In the MLE approach, the time series of the stock price is used to estimate firm volatility σ^F , which is assumed to be constant over the estimation period. The corresponding time series of firm value V_t^{F*} can be computed using equation (4.35).

One of the major drawbacks of these calibration methods (both non-linear system approach and the MLE approach) is that the level of leverage effect is assumed to be known. They could be estimated from the firm's balance sheet by transforming the firm's liabilities into one zero coupon debt. However, a firm's balance sheet and capital structure could be very complicated and transforming its liabilities could be hard, especially if it contains convertible debt. Even worse, the balance sheet information might not be available to the public.

As discussed in the previous sections, one approach to calibrate structural model is to use the volatility skew implied from the equity options. This approach does not require the knowledge of default boundary K . However, it requires that there must be traded equity options written on the firm's stock. Another limitation is that this approach gives the point-in-time market implied leverages, which are noisy when compared with the balance sheet leverages (see for example, Figure 4.17). Following the logic in Section 4.3, the levels of leverage effect in structural framework with warrants can be inferred using market information as shown in the discussion below.

4.5.2 Market Implied Leverage Effect Level

Let p be the dilution factor, i.e., $p \equiv M/(M + N)$, $\alpha^W \equiv \bar{V}^F/V_t^{F*}$ and β^W be the ratio of cash inflow from warrants exercise to default boundary, i.e., $\beta^W \equiv MK^W/K$. The warrant pricing equation (4.29) can be written as

$$\begin{aligned} X_t = e^{-r(T_w-t)} \int_{\log \alpha^W}^{\infty} \frac{V_t^{F*}}{N + M} & \left((e^x + \beta^W L) \Phi(d_1^W) - L e^{-r(\bar{T}-T_w)} \Phi(d_2^W) \right. \\ & - (e^x + \beta^W L)^{1-2\lambda} L^{2\lambda} \Phi(d_3^W) + e^{-r(\bar{T}-T_w)} L^{2\lambda-1} (e^x + \beta^W L)^{2-2\lambda} \Phi(d_4^W) \Big) g_{T_w-t}(x, \log L) dx \\ & - e^{-r(T_w-t)} \frac{\beta^W L}{M} V_t^{F*} G_{T_w-t}(\log \alpha^W, \log L), \end{aligned} \quad (4.37)$$

where

$$d_1^W = (-\log(L) + \log(\beta^W L + e^x) + (r + \sigma^{F^2}/2)(\bar{T} - T_w)) / (\sigma^F \sqrt{\bar{T} - T_w}), \quad (4.38)$$

$$d_2^W = d_1^W - \sigma^F \sqrt{\bar{T} - T_w}, \quad (4.39)$$

$$d_3^W = (\log(L) - \log(\beta^W L + e^x) + (r + \sigma^{F^2}/2)(\bar{T} - T_w)) / (\sigma^F \sqrt{\bar{T} - T_w}), \quad (4.40)$$

$$d_4^W = d_3^W - \sigma^F \sqrt{\bar{T} - T_w}. \quad (4.41)$$

Stock value given by equation (4.30) can be written as

$$\begin{aligned} S_t = e^{-r(T_w-t)} V_t^{F*} & \left(\int_{\log \alpha^W}^{\infty} \frac{1}{N + M} \left((e^x + \beta^W L) \Phi(d_1^W) - L e^{-r(\bar{T}-T_w)} \Phi(d_2^W) \right. \right. \\ & - (e^x + \beta^W L)^{1-2\lambda} L^{2\lambda} \Phi(d_3^W) + e^{-r(\bar{T}-T_w)} L^{2\lambda-1} (e^x + \beta^W L)^{2-2\lambda} \Phi(d_4^W) \Big) g_{T_w-t}(x, \log L) dx \\ & + \int_{\log L}^{\log \alpha^W} \frac{1}{N} (e^x \Phi(\tilde{d}_1^W) - L e^{-r(\bar{T}-T_w)} \Phi(\tilde{d}_2^W) - e^{x(1-2\lambda)} L^{2\lambda} \Phi(\tilde{d}_3^W) \\ & \left. + e^{-r(\bar{T}-T_w)} L^{2\lambda-1} e^{x(2-2\lambda)} \Phi(\tilde{d}_4^W) \Big) g_{T_w-t}(x, \log L) dx \right), \end{aligned} \quad (4.42)$$

where

$$\tilde{d}_1^W = (-\log(L) + x + (r + \sigma^F/2)(\bar{T} - T_W))/(\sigma^F \sqrt{\bar{T} - T_W}), \quad (4.43)$$

$$\tilde{d}_2^W = \tilde{d}_1^W - \sigma^F \sqrt{\bar{T} - T_W}, \quad (4.44)$$

$$\tilde{d}_3^W = (\log(L) - x + (r + \sigma^F/2)(\bar{T} - T_W))/(\sigma^F \sqrt{\bar{T} - T_W}), \quad (4.45)$$

$$\tilde{d}_4^W = \tilde{d}_3^W - \sigma^F \sqrt{\bar{T} - T_W}. \quad (4.46)$$

As in equation (4.19), we derive the relationship between the Black-Scholes implied volatility ν of warrant and other model parameters by matching the equation (4.37) with the warrant price using Black-Scholes formula. Let κ^W be the moneyness of the warrant, i.e., $\kappa^W S_t = K^W$. The relationship equation $X_t = BS(S_t, \nu)^5$ is

$$X_t = S_t(\Phi(l_1^W) - e^{-r(T_W-t)} \kappa^W \Phi(l_2^W)), \quad (4.47)$$

where

$$l_1^W = (-\log(\kappa^W) + (r + \nu^2/2)(T_W - t))/(\nu \sqrt{T_W - t}), \quad (4.48)$$

$$l_2^W = l_1^W - \nu \sqrt{T_W - t}. \quad (4.49)$$

After simplification, equation (4.47) becomes

$$\begin{aligned} & \left(p \int_{\log \alpha^W}^{\infty} ((e^x + \beta^W L) \Phi(d_1^W) - L e^{-r(\bar{T}-T_W)} \Phi(d_2^W) - (e^x + \beta^W L)^{1-2\lambda} L^{2\lambda} \Phi(d_3^W)) \right. \\ & \quad \left. + e^{-r(\bar{T}-T_W)} L^{2\lambda-1} (e^x + \beta^W L)^{2-2\lambda} \Phi(d_4^W)) g_{T_W-t}(x, \log L) dx - \beta^W L G_{T_W-t}(\log \alpha^W, \log L) \right) \\ & \div \left(p \int_{\log \alpha^W}^{\infty} ((e^x + \beta^W L) \Phi(d_1^W) - L e^{-r(\bar{T}-T_W)} \Phi(d_2^W) - (e^x + \beta^W L)^{1-2\lambda} L^{2\lambda} \Phi(d_3^W)) \right. \\ & \quad \left. + e^{-r(\bar{T}-T_W)} L^{2\lambda-1} (e^x + \beta^W L)^{2-2\lambda} \Phi(d_4^W)) g_{T_W-t}(x, \log L) dx + \right. \\ & \quad \frac{p}{1-p} \int_{\log L}^{\log \alpha^W} (e^x \Phi(\tilde{d}_1^W) - L e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_2^W) - e^{x(1-2\lambda)} L^{2\lambda} \Phi(\tilde{d}_3^W) \\ & \quad \left. + e^{-r(\bar{T}-T_W)} L^{2\lambda-1} e^{x(2-2\lambda)} \Phi(\tilde{d}_4^W)) g_{T_W-t}(x, \log L) dx \right) = \Phi(l_1^W) - e^{-r(T_W-t)} \kappa^W \Phi(l_2^W). \end{aligned} \quad (4.50)$$

Using the definition of \bar{V}^F , we can derive the relationship between α^W and β^W , which is given by the following equation

$$\begin{aligned} \beta^W L &= p((\alpha^W + \beta^W L) \Phi(d_{1,T_W}) - L e^{-r(\bar{T}-T_W)} \Phi(d_{2,T_W}) - (\alpha^W + \beta^W L)^{1-2\lambda} L^{2\lambda} \Phi(d_{3,T_W}) \\ & \quad + e^{-r(\bar{T}-T_W)} L^{2\lambda-1} (\alpha^W + \beta^W L)^{2-2\lambda} \Phi(d_{4,T_W})), \end{aligned} \quad (4.51)$$

⁵BS(*) is the Black-Scholes formula for European call option; it is a function of the share price S_t and the stock volatility ν .

where

$$d_{1,T_W} = (-\log(L) + \log(\alpha^W + \beta^W L) + (r + \sigma^F/2)(\bar{T} - T_W))/(\sigma^F \sqrt{\bar{T} - T_W}), \quad (4.52)$$

$$d_{2,T_W} = d_{1,T_W} - \sigma^F \sqrt{\bar{T} - T_W}, \quad (4.53)$$

$$d_{3,T_W} = (\log(L) - \log(\alpha^W + \beta^W L) + (r + \sigma^F/2)(\bar{T} - T_W))/(\sigma^F \sqrt{\bar{T} - T_W}), \quad (4.54)$$

$$d_{4,T_W} = d_{3,T_W} - \sigma^F \sqrt{\bar{T} - T_W}. \quad (4.55)$$

When solving the implicit equation, we set the initial guess for β^W as $p(1 - L)/(L(1 - p))^6$.

For a given α^W , the corresponding β^W could be computed by using equation (4.51). The moneyness of warrant κ^W can be computed by using equation below

$$\begin{aligned} \beta^W L = \kappa^W & \left(e^{-r(T_W - t)} \left(p \int_{\log \alpha^W}^{\infty} ((e^x + \beta^W L) \Phi(d_1^W) - L e^{-r(\bar{T} - T_W)} \Phi(d_2^W)) \right. \right. \\ & - (e^x + \beta^W L)^{1-2\lambda} L^{2\lambda} \Phi(d_3^W) + e^{-r(\bar{T} - T_W)} L^{2\lambda-1} (e^x + \beta^W L)^{2-2\lambda} \Phi(d_4^W)) g_{T_W-t}(x, \log L) dx \\ & + \frac{p}{1-p} \int_{\log L}^{\log \alpha^W} (e^x \Phi(\tilde{d}_1^W) - L e^{-r(\bar{T} - T_W)} \Phi(\tilde{d}_2^W) - e^{x(1-2\lambda)} L^{2\lambda} \Phi(\tilde{d}_3^W) \\ & \left. \left. + e^{-r(\bar{T} - T_W)} L^{2\lambda-1} e^{x(2-2\lambda)} \Phi(\tilde{d}_4^W)) g_{T_W-t}(x, \log L) dx \right). \end{aligned} \quad (4.56)$$

Figure 4.6 shows the relationship between the moneyness κ^W and the model implied volatility ν under the first passage time framework with warrants. When comparing to the results in Section 4.3 (see Figures 4.1), we can see that with warrants in the capital structure, the model implied volatility is slightly smaller than that of modeling without warrants. This is because the fixed firm volatility is distributed among shares, warrants and debts in this model. Volatility of debt value is small⁷; adding warrants in the capital structure will decrease the proportion of volatility that distributes to common shares. With equations (4.51) and (4.56), the relationship between variables κ^W , α^W and β^W are given. Using equation (4.50) along with the relationships between κ^W , α^W and β^W , we can calibrate the structural model using the warrants' Black-Scholes implied volatility ν . Two implied volatilities are needed for calibration since there are two unknowns, which are the leverage L and firm volatility σ_t^F in equation (4.50). However, our model assumes that there is only one type of warrant outstanding, i.e., all warrants have the same strike and maturity date⁸. To overcome this parameter issue, we first calibrate the leverage L using options data as in Section 4.3. Given the estimated L from options data, we infer the firm volatility σ_t^F using equation (4.50).

A similar discussion of deriving the relationship between warrant moneyness κ^W and the Black-Scholes implied volatility ν under Merton's framework is given in Section C.3.1. Figure 4.8 shows the κ^W and ν relationship under the Merton framework. It shows that the Black-Scholes implied volatility ν is less sensitive to model parameters leverage L and firm volatility σ^F under Merton's framework than under the first-passage time framework. Figure 4.9 shows the relationship between warrant moneyness w.r.t. stock price κ^W and warrant moneyness w.r.t. firm value α^W under Merton framework.

⁶We use the fact that $p = M/(N + M) \approx K^W M/(N * S_t + K^W M)$ and $L \approx K/(N * S_t + K)$.

⁷Volatility of debt value could be larger when the firm value V_t^F is close to default boundary K .

⁸This is commonly observed in practices. However, there are cases where firms issue multiple warrants.

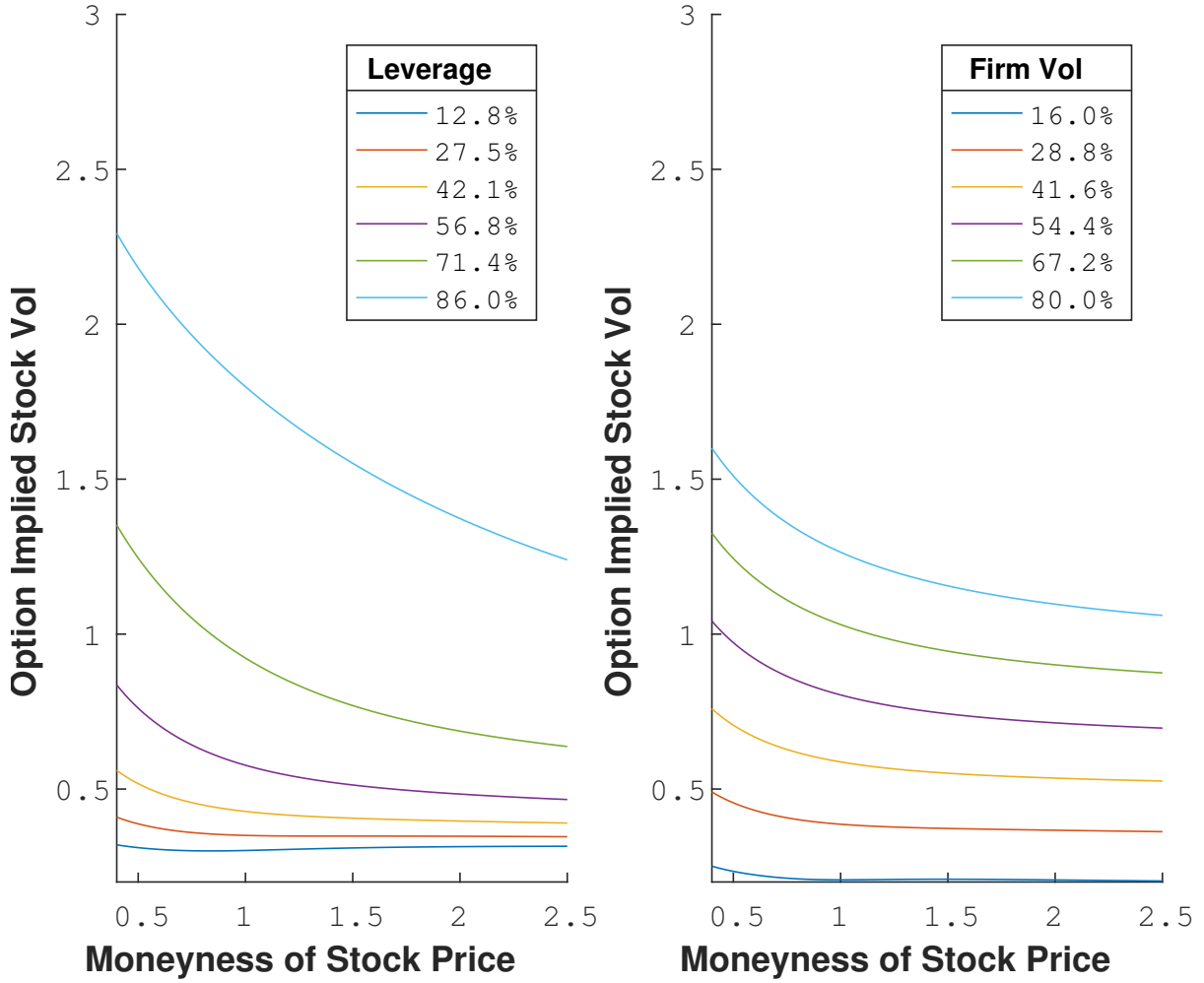


Figure 4.6: First-passage-time warrant model implied volatility skew - Moneyness of stock price κ versus implied stock volatility ν . Parameters input are below: $p = 20\%$, $r = 5\%$, $t = 0$, $T_w = 1$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^*e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^*e^{r(\bar{T}-t)}$).

4.5.3 Option Price of Firm with Capital Structure including Warrants

As discussed in Section 4.4, the exercise of warrants can potentially change the firm value, i.e., there is a dilution effect upon warrant exercise. This would impact the option price of a firm with warrants in its capital structure. Let T_o be the maturity date of options and assume that $T_o \leq T_w \leq \bar{T}$. Let \bar{V}_O^F be the threshold value such that, $S(\bar{V}_O^F, \sigma^F) = K^O$, where the stock price function is given by equation (4.30). Let $\bar{V}_O^F = \alpha V_t^F$ and $K^O = \kappa S_t$. The call option price

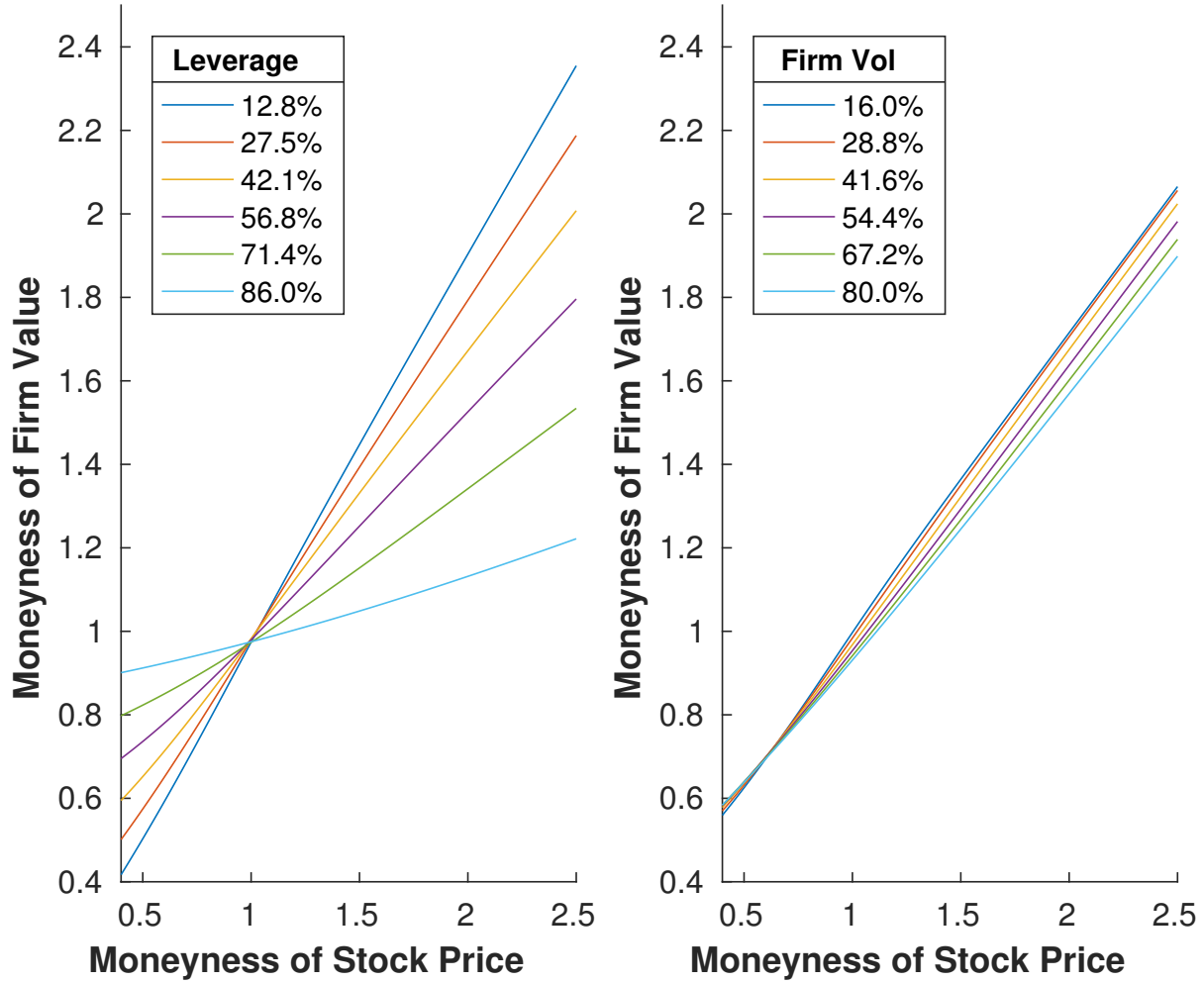


Figure 4.7: First-passage-time warrant model - Moneyness of stock price κ versus Moneyness of firm value α . Parameters input are below: $p = 20\%$, $r = 5\%$, $t = 0$, $T_w = 1$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

under the first passage time framework is

$$\begin{aligned}
 C_t &= e^{-r(T_0-t)} E_Q \left[(S(V_{T_0}^{F*}, \sigma^F) - K^0) I_{\{V_{T_0}^F \geq \bar{V}_0^F, m_{T_0}^{V^F} > K\}} \right] \\
 &= e^{-r(T_0-t)} \int_{\log \alpha}^{\infty} e^{-r(T_w-T_0)} V_t^{F*} e^y \left(\frac{I_1^C(y)}{N+M} + \frac{I_2^C(y)}{N} \right) g_{T_0-t}(y, \log L) dy \\
 &\quad - e^{-r(T_0-t)} \kappa S_t G_{T_0-t}(\log \alpha, \log L),
 \end{aligned} \tag{4.57}$$

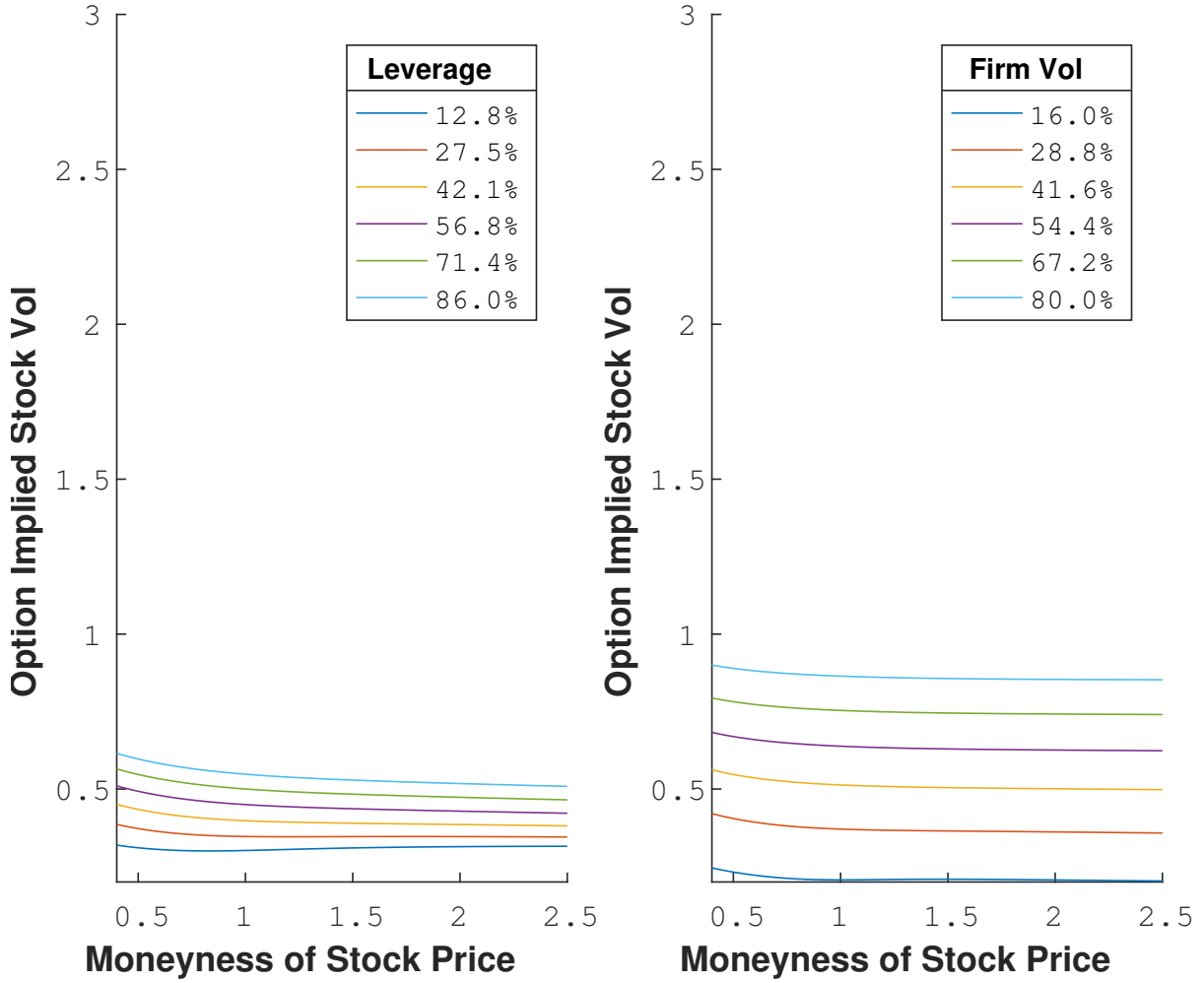


Figure 4.8: Merton warrant model implied volatility skew - Moneyness of stock price κ versus implied stock volatility v . Parameters input are below: $p = 20\%$ $r = 5\%$, $t = 0$, $T_w = 1$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

where

$$\begin{aligned}
 I_1^C(y) = & \int_{\log \alpha^W - y}^{\infty} ((e^x + \beta^W L e^{-y}) \Phi(d_{T_o,1}^W) \\
 & - L e^{-y} e^{-r(\bar{T}-T_w)} \Phi(d_{T_o,2}^W) - (e^x + \beta^W L e^{-y})^{1-2\lambda} (L e^{-y})^{2\lambda} \Phi(d_{T_o,3}^W) \\
 & + e^{-r(\bar{T}-T_w)} (L e^{-y})^{2\lambda-1} (e^x + \beta^W L e^{-y})^{2-2\lambda} \Phi(d_{T_o,4}^W)) g_{T_w-T_o}(x, \log L - y) dx \quad (4.58)
 \end{aligned}$$

$$\begin{aligned}
 I_2^C(y) = & \int_{\log L - y}^{\log \alpha^W - y} (e^x \Phi(\tilde{d}_{T_o,1}^W) - L e^{-y} e^{-r(\bar{T}-T_w)} \Phi(\tilde{d}_{T_o,2}^W) - e^{x(1-2\lambda)} (L e^{-y})^{2\lambda} \Phi(\tilde{d}_{T_o,3}^W) \\
 & + e^{-r(\bar{T}-T_w)} (L e^{-y})^{2\lambda-1} e^{x(2-2\lambda)} \Phi(\tilde{d}_{T_o,4}^W)) g_{T_w-T_o}(x, \log L - y) dx \quad (4.59)
 \end{aligned}$$

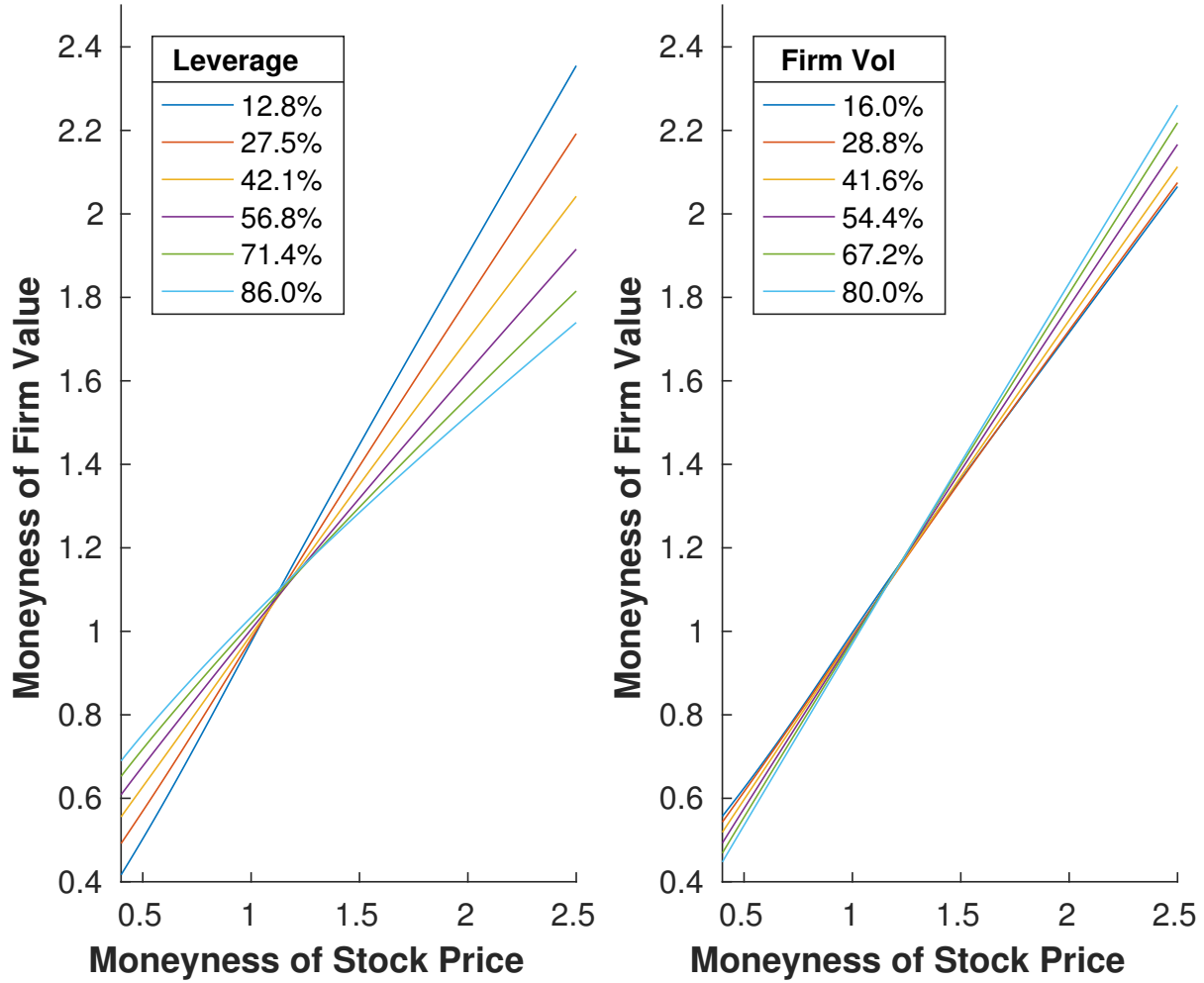


Figure 4.9: Merton warrant model - Moneyness of stock price κ versus Moneyness of firm value α . Parameters input are below: $p = 20\%$ $r = 5\%$, $t = 0$, $T_w = 1$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

and

$$d_{T_o,1}^W = (-\log(L) + y + \log(\beta^W L e^{-y} + e^x) + (r + \sigma^{F^2}/2)(\bar{T} - T_w)) / (\sigma^F \sqrt{\bar{T} - T_w}), \quad (4.60)$$

$$d_{T_o,2}^W = d_{T_o,1}^W - \sigma^F \sqrt{\bar{T} - T_w}, \quad (4.61)$$

$$d_{T_o,3}^W = (\log(L) - y - \log(\beta^W L e^{-y} + e^x) + (r + \sigma^{F^2}/2)(\bar{T} - T_w)) / (\sigma^F \sqrt{\bar{T} - T_w}), \quad (4.62)$$

$$d_{T_o,4}^W = d_{T_o,3}^W - \sigma^F \sqrt{\bar{T} - T_w}, \quad (4.63)$$

and

$$\tilde{d}_{T_o,1}^W = (-\log(L) + y + x + (r + \sigma^{F^2}/2)(\bar{T} - T_W))/(\sigma^F \sqrt{\bar{T} - T_W}), \quad (4.64)$$

$$\tilde{d}_{T_o,2}^W = \tilde{d}_{T_o,1}^W - \sigma^F \sqrt{\bar{T} - T_W}, \quad (4.65)$$

$$\tilde{d}_{T_o,3}^W = (\log(L) - y - x + (r + \sigma^{F^2}/2)(\bar{T} - T_W))/(\sigma^F \sqrt{\bar{T} - T_W}), \quad (4.66)$$

$$\tilde{d}_{T_o,4}^W = \tilde{d}_{T_o,3}^W - \sigma^F \sqrt{\bar{T} - T_W}, \quad (4.67)$$

and S_t , α^W and β^W are calculated by using equations (4.42), (4.56) and (4.51), respectively, when κ^K , L , and σ^F are given. As in the previous sections, we will show that α is a function of the moneyness κ . So that the call option value given by equation (4.57) is eventually a function of L , σ^F and V_t^{F*} .

By matching the call option value given by equation (4.57) and the Black-Scholes call option price given by equation (4.16), the relationship between the Black-Scholes implied volatility ν and the model parameters L and σ^F is

$$\begin{aligned} & \int_{\log \alpha}^{\infty} e^y \left(p I_1^C(y) + \frac{p}{1-p} I_2^C(y) \right) g_{T_o-t}(y, \log L) dy - e^{-r(T_o-t)} \kappa G_{T_o-t}(\log \alpha, \log L) \left(p I_1^W + \frac{p}{1-p} I_2^W \right) \\ &= (\Phi(l_1) - \kappa e^{-r(T_o-t)} \Phi(l_2)) \left(p I_1^W + \frac{p}{1-p} I_2^W \right), \end{aligned} \quad (4.68)$$

where

$$\begin{aligned} I_1^W &= \int_{\log \alpha^W}^{\infty} ((e^x + \beta^W L) \Phi(d_1^W) - L e^{-r(\bar{T}-T_W)} \Phi(d_2^W) - (e^x + \beta^W L)^{1-2\lambda} L^{2\lambda} \Phi(d_3^W) \\ &\quad + e^{-r(\bar{T}-T_W)} L^{2\lambda-1} (e^x + \beta^W L)^{2-2\lambda} \Phi(d_4^W)) g_{T_W-t}(x, \log L) dx \end{aligned} \quad (4.69)$$

$$\begin{aligned} I_2^W &= \int_{\log L}^{\log \alpha^W} (e^x \Phi(\tilde{d}_1^W) - L e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_2^W) - e^{x(1-2\lambda)} L^{2\lambda} \Phi(\tilde{d}_3^W) \\ &\quad + e^{-r(\bar{T}-T_W)} L^{2\lambda-1} e^{x(2-2\lambda)} \Phi(\tilde{d}_4^W)) g_{T_W-t}(x, \log L) dx, \end{aligned} \quad (4.70)$$

and $I_1^C(y)$, $I_2^C(y)$, l_1 , and l_2 are defined by equations (4.58), (4.59), (4.17) and (4.18), respectively.

Similar as in Section 4.3, the relationship between κ and α is given by the definition of \bar{V}_O^F ,

i.e., $S_{T_0}(\bar{V}_0^F) = K^O = \kappa S_t(V_t^F)$. After simplification, this is

$$\begin{aligned}
\kappa = & e^{r(T_0-t)} \alpha \left(p \int_{\log \alpha^W - \log \alpha}^{\infty} ((e^x + \beta^W \frac{L}{\alpha}) \Phi(\bar{d}_{T_0,1}^W) \right. \\
& - \frac{L}{\alpha} e^{-r(\bar{T}-T_W)} \Phi(\bar{d}_{T_0,2}^W) - (e^x + \beta^W \frac{L}{\alpha})^{1-2\lambda} (\frac{L}{\alpha})^{2\lambda} \Phi(\bar{d}_{T_0,3}^W) \\
& + e^{-r(\bar{T}-T_W)} (\frac{L}{\alpha})^{2\lambda-1} (e^x + \beta^W \frac{L}{\alpha})^{2-2\lambda} \Phi(\bar{d}_{T_0,4}^W)) g_{T_W-T_0}(x, \log L - \log \alpha) dx \\
& + \frac{p}{1-p} \int_{\log L - \log \alpha}^{\log \alpha^W - \log \alpha} (e^x \Phi(\hat{d}_{T_0,1}^W) - \frac{L}{\alpha} e^{-r(\bar{T}-T_W)} \Phi(\hat{d}_{T_0,2}^W) - e^{x(1-2\lambda)} (\frac{L}{\alpha})^{2\lambda} \Phi(\hat{d}_{T_0,3}^W) \\
& + e^{-r(\bar{T}-T_W)} (\frac{L}{\alpha})^{2\lambda-1} e^{x(2-2\lambda)} \Phi(\hat{d}_{T_0,4}^W)) g_{T_W-T_0}(x, \log L - \log \alpha) dx \Big) \\
& \div \left(p I_1^W + \frac{p}{1-p} I_2^W \right), \tag{4.71}
\end{aligned}$$

where

$$\bar{d}_{T_0,1}^W = (-\log(L) + \log(\alpha) + \log(\beta^W \frac{L}{\alpha} + e^x) + (r + \sigma^{F^2}/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}), \tag{4.72}$$

$$\bar{d}_{T_0,2}^W = \bar{d}_{T_0,1}^W - \sigma^F \sqrt{\bar{T} - T_W}, \tag{4.73}$$

$$\bar{d}_{T_0,3}^W = (\log(L) - \log(\alpha) - \log(\beta^W \frac{L}{\alpha} + e^x) + (r + \sigma^{F^2}/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}), \tag{4.74}$$

$$\bar{d}_{T_0,4}^W = \bar{d}_{T_0,3}^W - \sigma^F \sqrt{\bar{T} - T_W}, \tag{4.75}$$

and

$$\hat{d}_{T_0,1}^W = (-\log(L) + \log(\alpha) + x + (r + \sigma^{F^2}/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}), \tag{4.76}$$

$$\hat{d}_{T_0,2}^W = \hat{d}_{T_0,1}^W - \sigma^F \sqrt{\bar{T} - T_W}, \tag{4.77}$$

$$\hat{d}_{T_0,3}^W = (\log(L) - \log(\alpha) - x + (r + \sigma^{F^2}/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}), \tag{4.78}$$

$$\hat{d}_{T_0,4}^W = \hat{d}_{T_0,3}^W - \sigma^F \sqrt{\bar{T} - T_W}, \tag{4.79}$$

and I_1^W and I_2^W are defined by equations (4.69) and (4.70), respectively.

Figure 4.10 shows the relationship between the call option moneyness κ and the Black-Scholes implied volatility ν . When comparing these results with Figure 4.1⁹ which are the results of the model without warrants, we find that these two figures are similar except that the existence of warrants slightly reduces the Black-Scholes implied volatility. Figure 4.11 shows the relationship between the call option moneyness w.r.t. stock price κ and the moneyness w.r.t. firm value α . When comparing this result with Figure 4.3 which is from the model without warrants, we find that the existence of warrants increases the sensitivity of model output α to model parameters L and σ^F . This is true because the model including warrants has more complicated capital structure. The option prices are assumed to contain more information (dilution effect) than model with simple capital structure.

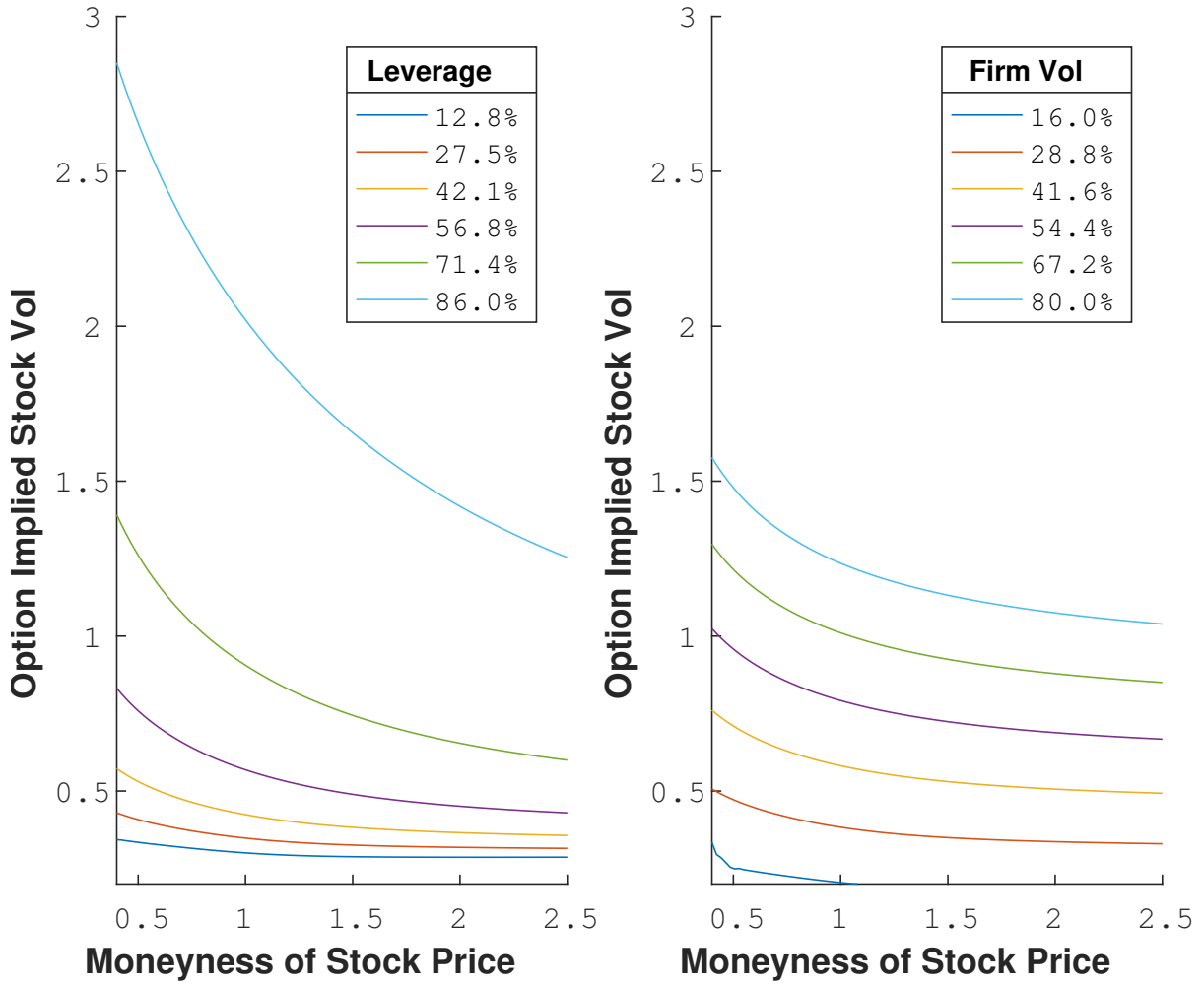


Figure 4.10: Call option under first-passage time model with warrant implied volatility skew - Moneyness of stock price κ versus implied stock volatility ν . Parameters input are below: $p = 20\%$, $r = 5\%$, $t = 0$, $T_O = 0.33$, $T_W = 1$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

Discussion on the call option price for firm with outstanding warrants under Merton's framework is given in Section C.3.2. Figure 4.12 shows the relationship between the call option moneyness κ and the Black-Scholes implied volatility ν under Merton's framework with warrants in a firm's capital structure. Figure 4.13 shows the relationship between the call option moneyness w.r.t. stock price κ and the moneyness w.r.t. firm value α under the Merton's framework with warrants. When comparing the results of the first-passage time framework and that of the Merton's framework, we find that the model output under the first-passage-time framework is more sensitive to model parameters (such as L and σ^F) than that of Merton's framework.

⁹Model parameters are the same for both figures.

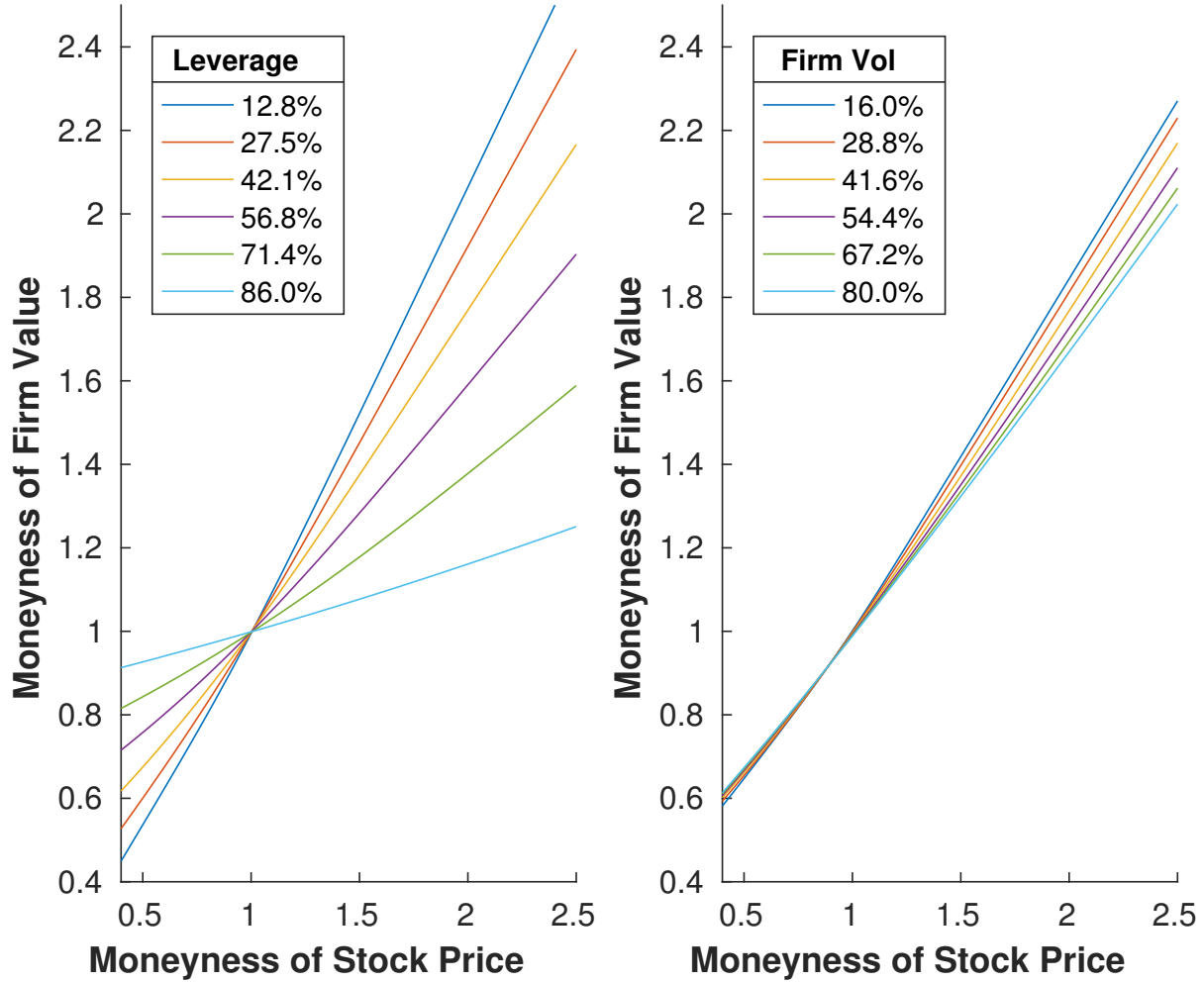


Figure 4.11: Call option under first-passage time model with warrant - Moneyness of stock price κ versus Moneyness of firm value α . Parameters input are below: $p = 20\%$ $r = 5\%$, $t = 0, T_O = 0.33, T_W = 1, \bar{T} = 5, L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

4.6 Model Performance - Case Studies

In this section, we test our models by using historical data. We collect and filter data following the empirical studies in [15]. For completeness, we detail the data collection process in this section.

4.6.1 Data Overview

Options and common shares data are collected from the Security Price file and the Option Price file from Ivy DB OptionMetrics. The data is from January 1996 to April 2016. For this case studies, we only focus on four companies, whose tickers are AAPL (Apple Inc.), IBM (International Business Machines Corporation), AEM (Agnico Eagle Mines Limited) and

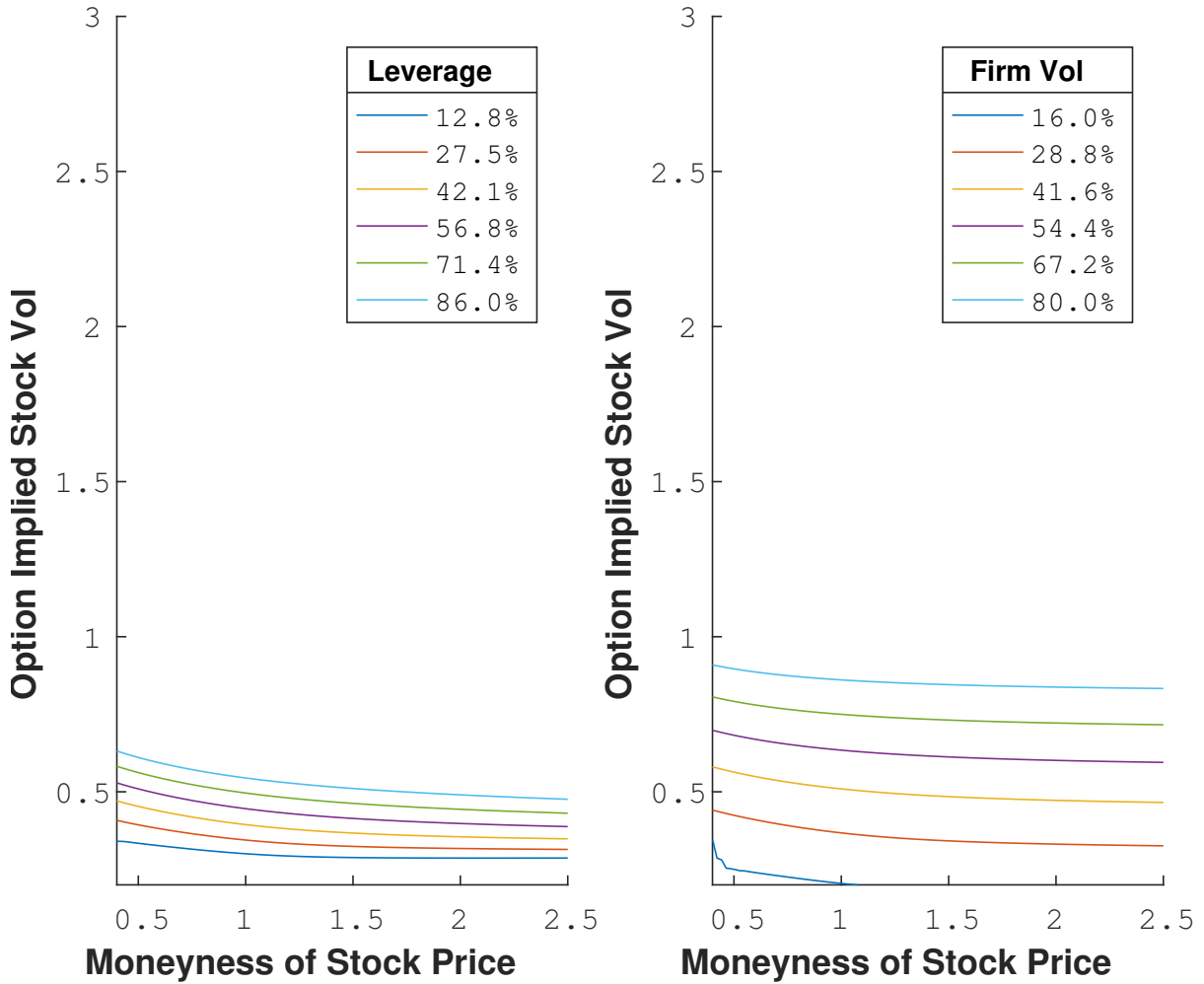


Figure 4.12: Call option under Merton model with warrant implied volatility skew - Moneyness of stock price κ versus implied stock volatility ν . Parameters input are below: $p = 20\%$ $r = 5\%$, $t = 0$, $T_O = 0.33$, $T_W = 1$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^* e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^* e^{r(\bar{T}-t)}$).

FNV (Franco-Nevada Corporation). The four companies are large cap companies with equity options traded on major exchanges.

From the Security Price file, we download CUSIP, the unique Security ID and Close Price for each date. From the Option Price file, we download the CUSIP; Strike Price; Expiration Date of the option; Call/Put Flag; Best Bid, the highest closing bid price across all exchanges on which the option trades; Best Offer, the lowest closing ask price across all exchanges on which the option trades; Volume, the total volume for the option; and Open interest for each date. Option data are filtered based on the following rule: the date on which the option last traded is not missing, open interest is positive, bid price is positive and is strictly smaller than offer price, and the Volume is positive. We use mid price, which is the average of Best Offer and Best Bid, as the option close price for each date.

The stock dividend data is downloaded from CRSP (the Center for Research in Security

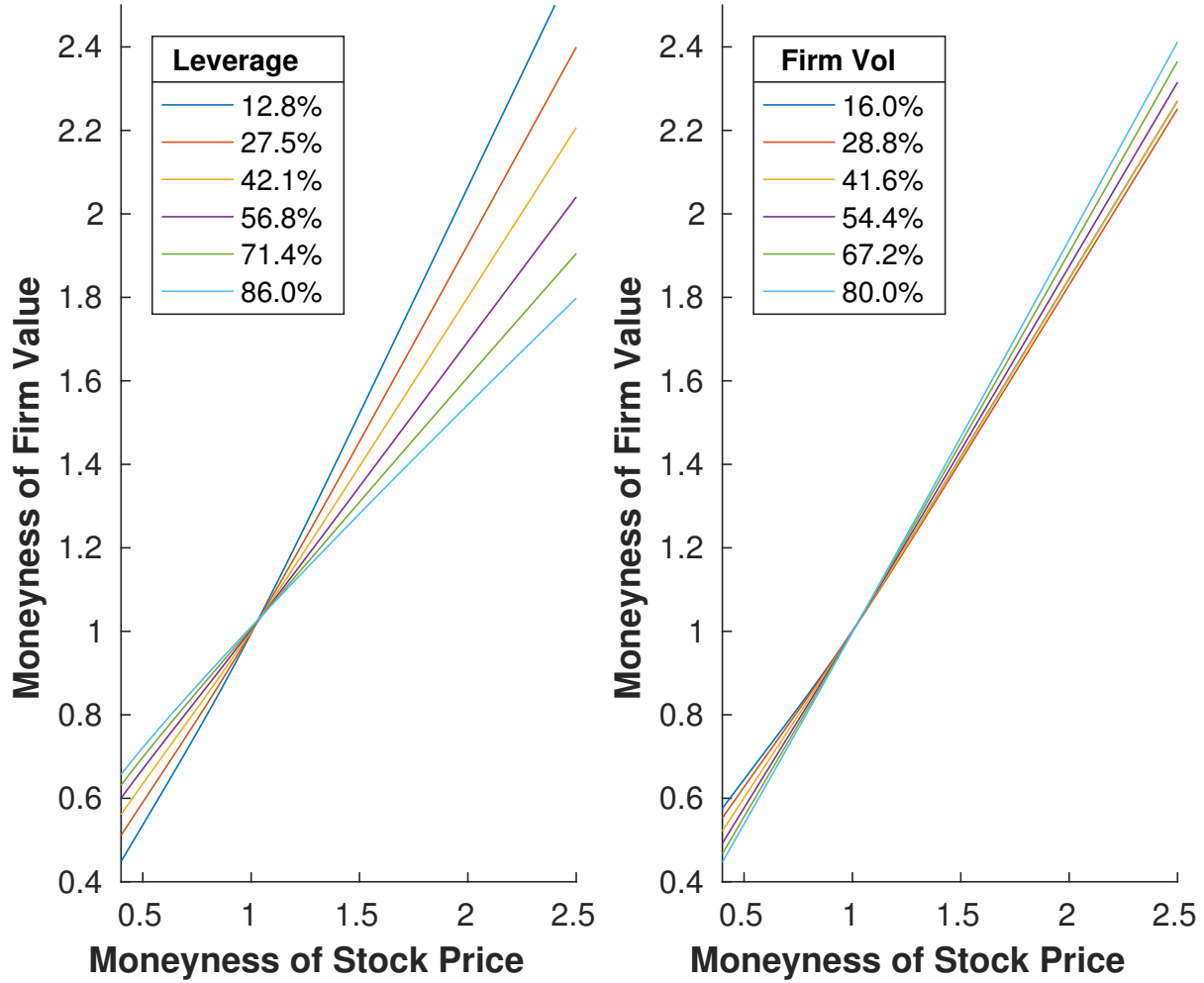


Figure 4.13: Call option under Merton model with warrant - Moneyness of stock price κ versus Moneyness of firm value α . Parameters input are below: $p = 20\%$, $r = 5\%$, $t = 0$, $T_O = 0.33$, $T_W = 1$, $\bar{T} = 5$, $L = 38.5\%$, and $\sigma^F = 30\%$ (The range of parameter inputs L and κ are chosen following Geske, Subrahmanyam and Zhou[15]. The leverage L^* in their paper is approximately equal to $L^*e^{r(\bar{T}-t)}$ in ours, i.e., $L = L^*e^{r(\bar{T}-t)}$).

Price). We downloaded CUSIP; Closing Price, Declaration Date, the date the board declares a distribution; Record Date, the date on which the stockholder must be registered as holder to receive a particular distribution; and Payment Date, the date upon which distributions are made. Since single-name exchange-traded options are American style. In order to implement our models, which assume European style options, we only keep options data for which the underlying stock pays no dividend during the remaining time to expiration. This allows the options to be treated as quasi-European calls because it is never optimal to exercise an American call option prior to maturity when underlying stock pays no dividend. Moreover, we removed options data when the arbitrage bounds are violated, i.e., $C \leq S - Ke^{-rtT}$.

Annual balance sheet information are obtained from Compustat data. We merge the data with option data by using CUSIP and year. We follow the maturity buckets defined in [15]. Debt maturing at the end of first year includes LCT, the total current liability; DD1, debt

due in one year not included in current liability, AEDI, accrued expense and deferred income; DC, deferred charges; TXDFED deferred federal tax; TXDFO, deferred foreign tax; TXDS, deferred state tax, and NP, notes payable. The total long-term debt is DLTT. Debt maturing at the end of second year, DD2; at the end of the third year, DD3; at end of the fourth year, DD4; and at the end of the fifth year, DD5. The debt imputed to be due in the seventh year includes DCLO, the capitalized lease obligation; DCS, the debt of the consolidated subsidiary; DFS, the finance subsidiary; DM, mortgage debt and other secured debt; DN, notes debt; LO, other liabilities; DD, debentures; CLG, contingent liabilities; DLTP, long-term debt tied to the prime rate; and $(DLTT - DD2 - DD3 - DD4 - DD5)$, all reported debt with maturity longer than five years.

The term structure of risk free rate is obtained by linearly interpolating the effective market yields using six month, one, two, three, five, ten and twenty year constant maturity bills and bonds from the Federal Reserve data for government securities.

We also follow the classification of volatility term and moneyness described in [15]. Time to expiration are classified by five ranges: 1. Very near term (21-40 days); 2. Near term (41-60 days); 3. Middle term (61-110 days); 4. Far term (111-170 days); 5. Very far term (171-365 days). Options expiring in less than 21 days or more than 365 days are omitted. Option moneyness is defined as the ratio of the strike price to the current stock price. The seven categories of option moneyness are defined as: 1. Very deep in-the-money (0.40-0.75); 2. Deep in-the-money (0.75-0.85); 3. In-the-money (0.85-0.95); 4. At-the-money (0.95-1.05); 5. Out-of-the-money (1.05-1.15); 6. Deep out-of-the-money (1.15-1.25); 7. Very deep out-of-the-money (1.25-2.50). Option moneyness with a ratio less than 0.4 or larger than 2.5 are removed because their light trading frequency and possible nonsynchronicity of trading.

4.6.2 Pricing European Call Options

In this section, we study the performance of the option pricing models described in Section 4.3; we compare the pricing errors of the option pricing model under the FPT framework (Section 4.3) with that of the Merton framework [15] and the traditional Black-Scholes model. More than 6 millions¹⁰ options data are downloaded for the four companies. After the filtering process described in Section 4.6.1, we tested the option pricing models with 367,296 market data. The results show that in general the option pricing model under FPT framework outperforms the Merton framework, and that the option pricing models under structural framework (both of Merton and FPT) outperform the traditional Black-Scholes option pricing model.

To test the performance of the models, we follow the empirical study in [15]. The Black-Scholes model is implemented by estimating the implied stock volatility of the at-the-money option for each expiration bucket¹¹ and using it to price all the options in the same expiration bucket on the same day. For structural models under Merton and FPT frameworks, the models are implemented by first estimating the implied leverage L and firm volatility σ^F using the prices of a set of three options¹² that has the shortest time to maturities within the same expira-

¹⁰In database Ivy DB OptionMetrics, there are 6,036,568 option data for AAPL, IBM, AEM and FNV within period from January 1996 to April 2016.

¹¹the expiration buckets are defined in Section 4.6.1

¹²Following the empirical study in [15], the three options contains one at-the-money option that is closest to the money, one out-of-the-money option that is closest to the money and one in-the-money option that is closest to

tion bucket. The leverage L and firm volatility σ^F are calibrated by minimizing¹³ the absolute pricing errors of the three options.¹⁴ The debt expiration \bar{T} is estimated by the duration of debt in balance sheet information¹⁵. The estimated L and σ^F for each expiration bucket are used to price all options within the same expiration bucket on the same date. Given the leverage L and firm volatility σ^F , the call option price under the First Passage Time framework is

$$C_t = \frac{e^{-r(T_0-t)} S_t}{\Phi(d_1) - L e^{-r(\bar{T}-t)} \Phi(d_2) - L^{2\lambda} \Phi(d_3) + e^{-r(\bar{T}-t)} L^{2\lambda-1} \Phi(d_4)} \\ \times \int_{\log \alpha}^{+\infty} (e^x \Phi(\tilde{d}_1) - e^{-r(\bar{T}-T_0)} L \Phi(\tilde{d}_2) - e^{x(1-2\lambda)} L^{2\lambda} \Phi(\tilde{d}_3) \\ + e^{-r(\bar{T}-T_0)-x(2\lambda-2)} L^{2\lambda-1} \Phi(\tilde{d}_4)) g_{T_0-t}(x, \log L) dx - e^{-r(T_0-t)} \kappa S_t G_{T_0-t}(\log \alpha, \log L), \quad (4.80)$$

where $d_1, d_2, d_3, d_4, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4$ and α are given by equations (4.3), (4.4), (4.5), (4.6), (4.12), (4.13), (4.14), (4.15) and (4.25), respectively. Given the leverage L and firm volatility σ^F , the call option price under the Merton framework is

$$C_t = \frac{S_t(\Phi_2(a_1, d_1^M; \rho_M) - e^{-r(\bar{T}-t)} L \Phi_2(a_2, d_2^M; \rho_M))}{\Phi(d_1^M) - e^{-r(\bar{T}-t)} L \Phi(d_2^M)} - e^{-r(T_0-t)} \kappa^M S_t \Phi(a_2), \quad (4.81)$$

where $d_1^M, d_2^M, \rho_M, a_1$, and a_2 are given by equations (C.14), (C.15), (C.17), (C.18), and (C.19), respectively.

Figure 4.14 illustrate the calibration approach used in this section. The cross markers are the Black-Scholes implied volatilities from the option prices observed from the market. Specifically, the red cross markers are the three market implied stock volatility used to calibrate the structural model for both the FPT framework and the Merton framework. The blue line is the calibrated volatility skew using the option pricing model under the FPT framework while the green line is output from the option pricing model under the Merton framework. In most cases, output of the option pricing model under the FPT framework fit the market implied volatility skew better than that under the Merton framework. This is in line with the discussion in Section 4.3 that the FPT framework can generate a larger range of volatility skews than the Merton framework. However, in some cases, the market implied volatility skew can be too steep where neither the Merton framework nor the FPT framework could generate a curve that is steep enough to fit the market volatility skew. Figure 4.15 illustrates an example of such case; it shows that the volatility skew generated by structural models are too flat when comparing to the market volatility skew. Figure 4.16 shows the calibration results using two option prices, with one at-the-money option that is closest to the money and one deepest in-the-money option. It shows that this calibration approach (using two options) improves the performance of model under FPT framework while the improvement of model under the Merton framework is limited. All of the figures (Figure 4.14, Figure 4.15, and Figure 4.16) show that structural models perform better in pricing in-the-money options than out-of-the-money options.

the money.

¹³Matlab optimizer *fmincon* is used in our implementation.

¹⁴We tested the calibration optimizer using simulated data with leverage range from 0.1 to 0.99, and firm value volatility range from 10% to 100%. In most of the cases, the numerical errors are less than 0.1% with the largest error of 2%.

¹⁵Debt duration is calculated using the treasure yields as discount rates as in [15]. Please see Section 4.3 for detailed discussion

Table 4.2: Overview of Model Performances - This table reports the average pricing errors (in percentage) of pricing models. BS Errors is the average pricing error of the Black-Scholes Model. Merton Errors is the average pricing error of option pricing model under the Merton structural framework. FPT Errors is the average pricing error of option pricing model under the First Passage Time structural framework.

Count	BS Errors %	Merton Errors %	FPT Errors %
367296	18.22	14.19	13.83

As the summary of model performance, we show some statistics from our case studies. In this section, the pricing error is calculated as

$$Error = \frac{\|Price_{Model} - Price_{Market}\|}{Price_{Market}}, \quad (4.82)$$

where $Price_{Model}$ is the output from the pricing model and $Price_{Market}$ is the market mid-price. Table 4.2 confirms the findings in [15] that the option pricing models under structural framework outperform the traditional Black-Scholes model. Moreover, the results show that on average the option pricing model under FPT structural framework outperforms the option pricing model under Merton structural framework.

Table 4.3: Pricing Errors Grouped by Option Moneyness - This table reports the average pricing error (in percentage) grouped by option moneyness. Detailed categorization of option moneyness is describe in Section 4.6.1. BS Errors is the average pricing error of the Black-Scholes Model. Merton Errors is the average pricing error of option pricing model under the Merton structural framework. FPT Errors is the average pricing error of the option pricing model under the First Passage Time structural framework.

Option Moneyness	Count	BS Errors %	Merton Errors %	FPT Errors %
Very deep in-the-money	40910	1.00	0.60	0.46
Deep in-the-money	35227	1.70	1.01	0.79
In-the-money	52970	2.06	1.52	1.19
At-the-Money	75556	4.32	4.04	3.66
Out-of-the-money	61064	18.28	15.71	15.50
Deep Out-of-the-money	38314	32.76	28.03	28.69
Very deep Out-of-the-money	63255	59.84	43.17	41.87

Table 4.3 summarizes the average pricing errors of the three models grouped by moneyness. The results show that for all option moneyness, the option pricing model under structural frameworks (both Merton and FPT) have smaller pricing errors than that of the Black-Scholes model. Moreover, option pricing model under FPT framework outperforms the Merton framework except for the Deep Out-of-the-money bucket. Another observation is that the pricing errors for in-the-money options is much smaller than that of the out-of-the-money options for all three models. Table 4.4 reports the average pricing errors of the three models grouped by

Table 4.4: Pricing Errors Grouped by Option Maturity Term - This table reports the average pricing error (in percentage) grouped by option maturity term. Detailed categorization of option maturity term is describe in Section 4.6.1. BS Errors is the average pricing error of the Black-Scholes Model. Merton Errors is the average pricing error of option pricing model under the Merton structural framework. FPT Errors is the average pricing error of the option pricing model under the First Passage Time structural framework.

Option Maturity Term	Count	BS Errors %	Merton Errors %	FPT Errors %
Very far term	66848	17.93	8.63	6.10
Far term	47066	18.93	8.72	7.87
Middle term	67409	17.66	13.04	12.89
Near term	64326	17.40	15.72	16.19
Very near term	121647	18.86	19.18	19.66

option maturity terms. The results show that the pricing errors of the pricing models under structural frameworks are smaller than that of the Black-Scholes model except for options with very near term maturity. This results confirm the findings of empirical study in [15] that option pricing model under the structural framework performs better for options with longer maturity term. Moreover, the option pricing model under the FPT framework outperforms the Merton framework for options with all maturity terms except for near term and very near term. This evidence shows that the FPT framework, when compared with the Merton framework, can better explain the leverage effects embedded in option prices, especially for options with long time to maturity.

Table 4.5 reports the pricing errors of the three option pricing models grouped by leverage and option moneyness. In this analysis, the leverage is defined using the market implied leverage by using model under the FPT framework. Low leverage, mid leverage and high leverage are categorized if the market implied leverage $L \in [0, 0.25)$, $L \in [0.25, 0.5)$ and $L \in [0.5, 1)$, respectively. This results show that the option pricing model under FPT framework outperforms the Black-Scholes model and the option pricing model under Merton framework for most of the categories. Specifically, the results show that the pricing accuracy improvement of option pricing model under the FPT framework over the Black-Scholes model increases with leverage.

Figure 4.17 shows the relationship between the option implied leverages L , balance sheet leverage L^B and the share price of IBM. The balance sheet leverage is defined as $L^B \equiv K^B / (EQ_t + K^B)$, where K^B is the face value of the zero coupon debt converted from the firm's total liabilities¹⁶. It depicts the inverse relationship between the model implied leverages and the share price for both Merton and First Passage Time structural frameworks. Especially, the options implied leverage under First Passage Time framework is more sensitive to the share price change than that of the Merton framework. The correlation between the balance sheet leverage L^B and the option implied leverage under the FPT framework is 13.12%, which is higher than that under the Merton framework (1.42%). This results support the argument in the literatures that the equity option volatility skew contains insightful information about the firm's capital structure.

¹⁶We follow the discussion in Section 3.2.1 to convert the firm's total liabilities into a zero coupon debt.

Our results confirm the findings in [15] that the option pricing models under structural frameworks outperform the traditional Black-Scholes model. Especially, the option pricing models under structural framework perform better for options with longer time to maturity and for firm with higher implied leverage. Moreover, the results in our case study show that the option pricing model under the FPT framework outperforms the option pricing model under the Merton framework for most of the option categories. The improvement of the pricing errors of the FPT framework relative to the Merton framework increases as option time to maturity increases. This case study shows that the FPT structural model can better explain the equity option skew than the Merton framework, especially for options with long time to maturity.

4.6.3 Pricing Equity Call Warrants

In this section, we demonstrate the implementation of our warrant pricing models by a case study. We downloaded warrants data (last price and outstanding warrants numbers) from Bloomberg for company AEM. The warrants data of AEM is from June 2009 to December 2012. Options data follows the data cleaning process described in Section 4.6.1. For each date, we select the options that have the longest time to maturities. We download 836 observations for AEM. Options data and balance sheet information are merged to the warrants data on each date. After data cleaning for options data, we filtered out 229 observations for AEM in this case study.

In this case study, we compare the performance of five models in pricing warrants. The five models are the Black-Scholes model, the option pricing model under Merton framework, the option pricing model under the FPT framework, the warrant pricing model under the Merton framework and the warrant pricing model under the FPT framework, respectively. To test the performance of the pricing models, we use information from option market prices and balance sheet to calibrate model parameters, and price warrants using the calibrated parameters.

The detailed implementation of the models are described below. The Black-Scholes model is implemented by estimating the implied stock volatility using the at-the-money call option and this implied volatility is used to price the warrant (as a simple European call option) on the same date. For option pricing models under Merton and First Passage Time frameworks, warrants are treated as European call options; the models are implemented as in Section 4.6.2, i.e., by first estimating the implied leverage L and firm volatility σ^F using the prices of a set of three closest to the money options. The warrants are priced using equations (4.81) and (4.80) for the Merton and FPT frameworks, respectively. For the warrants pricing models under the Merton and FPT frameworks, the models are implemented by first estimating the firm leverage L and firm volatility σ^F use the prices of a set to three closest to the money options. Leverage L and firm volatility σ^F are calibrated by minimizing the absolute pricing errors of the three options. The estimated L and σ^F for each date are used to price warrants on the same date. Given the leverage L and firm volatility σ^F , the warrant price under the First Passage Time framework is given by equation

$$W_t = \frac{S_t I_1^W}{I_1^W + \frac{1}{1-p} I_2^W} - \frac{S_t \beta^W L}{p I_1^W + \frac{p}{1-p} I_2^W} G_{T_w-t}(\log \alpha^W, \log L), \quad (4.83)$$

where $I_1^W, I_2^W, d_1^W, d_2^W, d_3^W, d_4^W, \tilde{d}_1^W, \tilde{d}_2^W, \tilde{d}_3^W, \tilde{d}_4^W, \alpha^W$ and β^W are given by equations (4.69), (4.70), (4.38), (4.39), (4.40), (4.41), (4.43), (4.44), (4.45), (4.46), (4.56) and (4.51). For the Merton

framework, if leverage L and firm volatility σ^F are given, the warrant price is

$$W_t = \frac{S_t \tilde{I}_1^W}{\tilde{I}_1^W + \frac{1}{1-p} \tilde{I}_2^W} - \frac{S_t \beta^W L}{p \tilde{I}_1^W + \frac{p}{1-p} \tilde{I}_2^W} \Phi(-\log \alpha^W; -\mu_{T_W-t}, \sigma_{T_W-t}), \quad (4.84)$$

where

$$\begin{aligned} \tilde{I}_1^W &= \int_{\log \alpha^W}^{+\infty} \left((e^x + \beta^W L) \Phi(d_{1,T_W}^M) - Le^{-r(\bar{T}-T_W)} \Phi(d_{2,T_W}^M) \right) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \\ \tilde{I}_2^W &= \int_{-\infty}^{\log \alpha^W} \left(e^x \Phi(\tilde{d}_{1,T_W}^M) - Le^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_{2,T_W}^M) \right) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx, \end{aligned}$$

and d_{1,T_W}^M , d_{2,T_W}^M , \tilde{d}_{1,T_W}^M , \tilde{d}_{2,T_W}^M , α^W , and β^W are given by equations (C.25), (C.26), (C.28), (C.29), (C.31) and (C.30), respectively.

Figure 4.18 illustrate the implementation of warrant pricing models discussed in this section. The cross markers in red are the implied volatilities from option prices, which used to calibrate models parameters. The cross marker in light blue is the Black-Scholes implied volatilities from warrant prices. The implied volatility skews associated with different models are calculated using the Black-Scholes formula, i.e., warrants are priced at different strikes using the given parameters, and the warrant prices are used to calculate the implied volatilities using the Black-Scholes formula for a call option. Results in this example confirms the fact that the volatility skew implied by models under the First Passage Time framework is more flexible than that under the Merton framework. Moreover, the steepness of volatility skew¹⁷ implied by the warrant price model is smaller than the option pricing model under the structural framework. It is because the dilution effects of warrants are taken into account in the warrant pricing models. In this example, it can be found that the level volatility skew implied by our models do not match exactly with the market implied volatilities of options. The reason is because the volatility skews shown in Figure 4.18 is implied by warrant prices calculated by our models. For the options pricing models under Merton and FPT framework, the mismatch comes from the parameter differences between the warrant that we priced and that options that used to calibrate the models, for example, the difference in time to maturities and the risk free rate associated to different terms¹⁸. Figure 4.19 shows another example of implementation of the warrant pricing models. It shows that the market implied volatility of warrant is lower than what our models implied, i.e., our models overprice warrants in many instances of this case study. As a result, the Black-Scholes model outperforms our warrant pricing models in these instances.

Table 4.6 reports the average pricing errors of the five models in pricing warrants. The pricing error is defined in equation (4.82). The results of AEM case study show that models under FPT framework outperforms the models under Merton framework. Moreover, the warrant pricing models outperform the option pricing models in pricing warrants, i.e., warrant pricing models with dilution effects outperform those without dilution effects. However, the Black Scholes model outperforms warrant pricing models under structural framework. This results is mainly cause by the situation explained in Figure 4.19. The reason may come from

¹⁷The volatility skew here means the volatility skew implied by the warrant price.

¹⁸The results of option pricing model under Merton framework is sensitive to risk free rate change.

the long maturity term and illiquidity of the warrant. Due the small sample size of this case study, the results given in this section are not conclusive.

4.7 Conclusion

In this paper we derive the close-form formula for European option price under the First Passage Time (FPT) structural framework and extended the calibration method of [18] and [15] to the FPT framework. We show that the model implied volatility skew under FPT framework is much more flexible than that under Merton framework.

Following [8], we extended the FPT structural framework to include warrants into a firm's capital structure. Moreover, we derive the European call option price under the warrant extended capital structure model under both Merton and FPT frameworks. Also, we proposed a calibration method for the warrant extended capital structure model under both Merton and FPT frameworks. As a results, warrants can be priced under a unified framework by using information from European call options.

Using a case study with more than thirty thousands options data we show that the option pricing model under FPT framework significantly outperforms the option pricing model under the Merton framework. The results of the case study confirm the empirical study in [15] that the option pricing models under structural frameworks outperform the Black-Scholes model, i.e., the equity option prices contain insightful information of a firm's capital structure. Moreover, our case study indicates that the FPT framework could better capture the embedded leverage information from options than the Merton framework.

A case study using AEM's warrants data demonstrate the implementation of our warrant pricing models under both Merton and FPT structural frameworks.

Future research will include: Performing an empirical study for the option pricing models by using more options data; performing empirical study for the warrant pricing model; studying the option and warrant pricing model under FPT framework by assuming different firm value dynamics (e.g., CEV process); studying the pricing of CDS and risky bonds under FPT framework with calibration using equity options; and studying the impact of liquidity in pricing warrants.

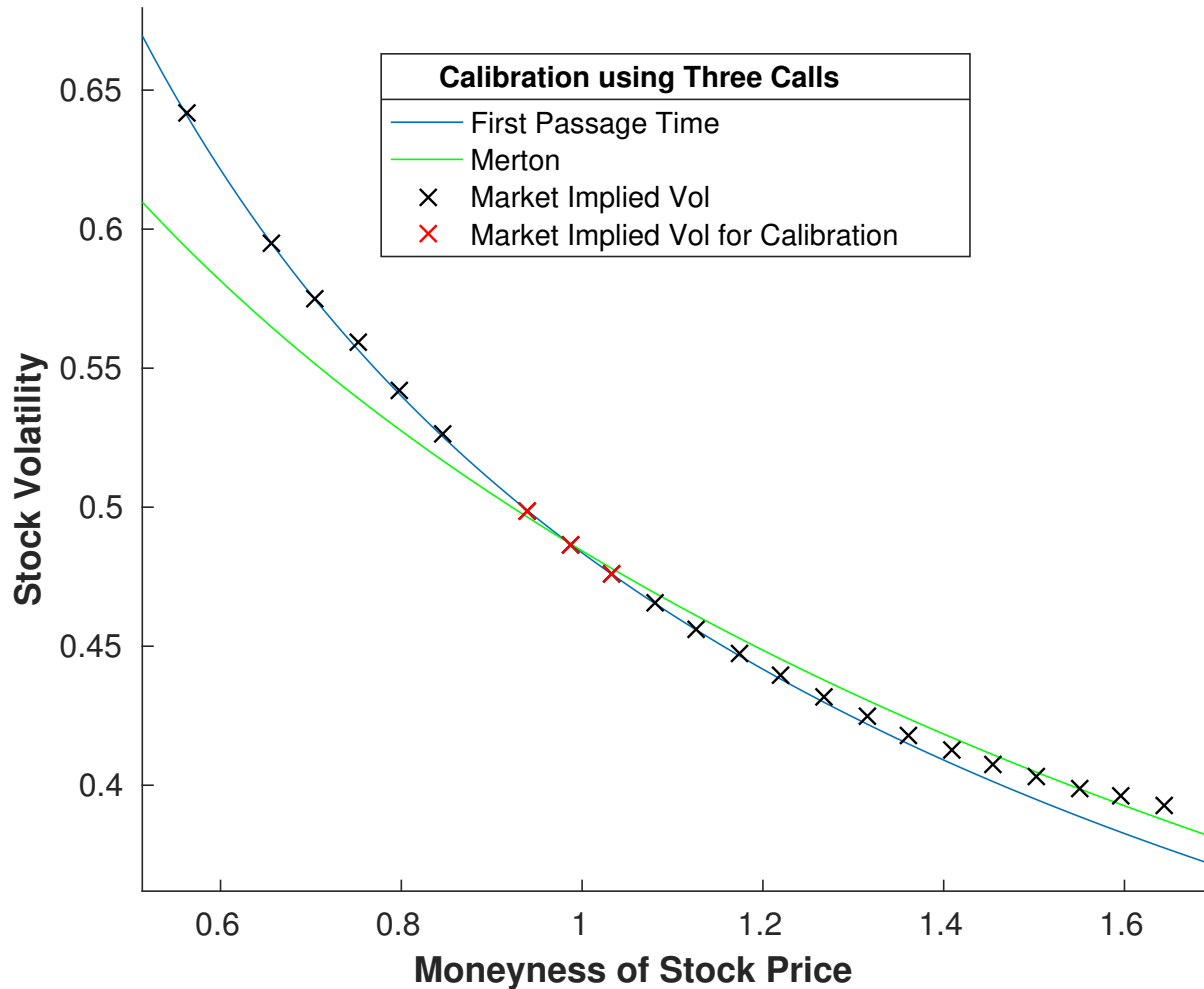


Figure 4.14: Fitting Market Implied Volatility Skew using Three Options - The cross markers are the Black-Scholes implied volatilities from the option prices observed from the market. Especially, the red cross markers are the three market implied stock volatility used to calibration the structural model for both the First Passage Time framework and the Merton framework. The blue line is the calibrated volatility skew using the option pricing model under the First Passage Time framework while the green line is output from the option pricing model under the Merton framework. The market data is the mid-price of call options on Apple's share on date 2009-03-25. The options' time to maturity is 206 days.

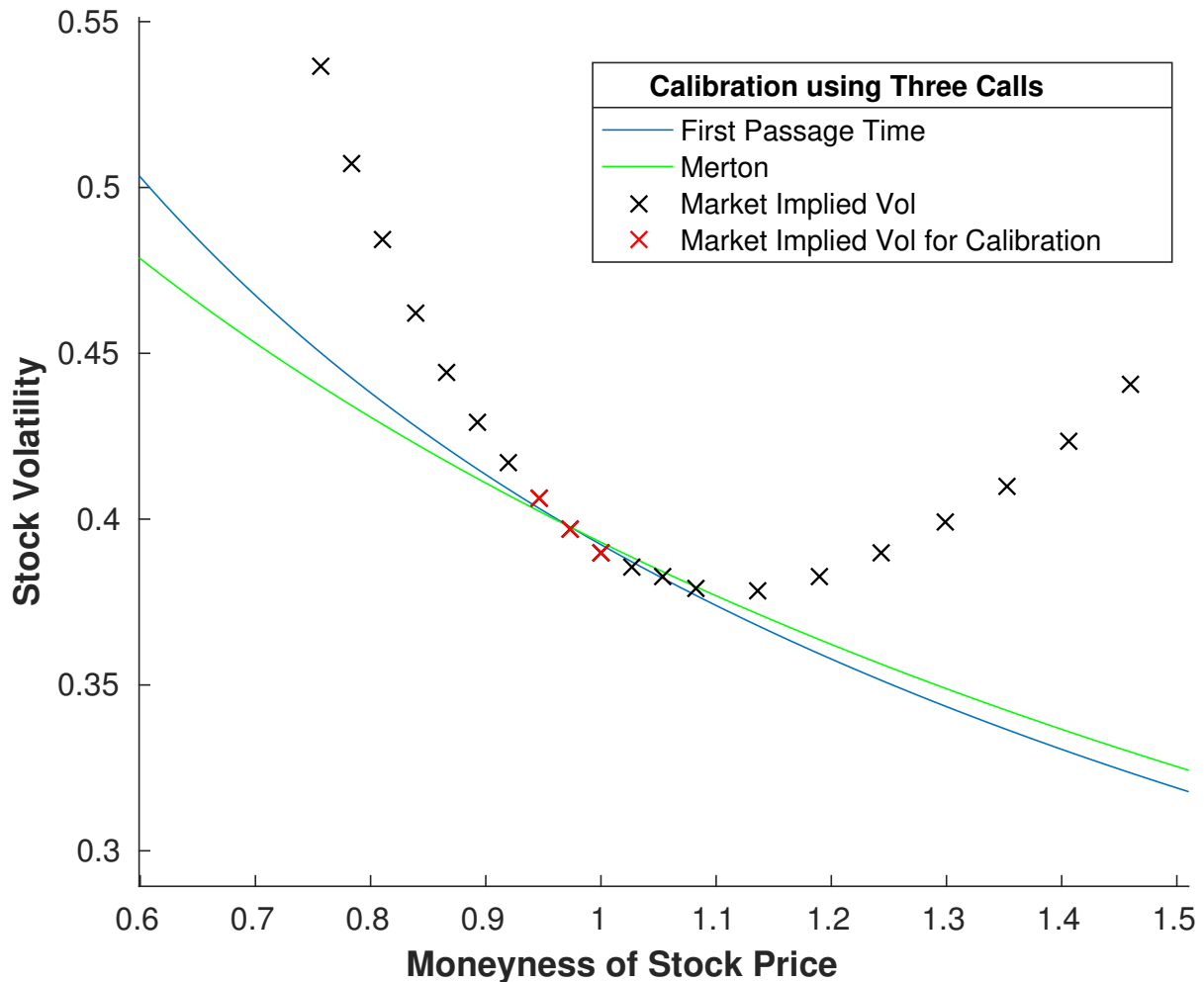


Figure 4.15: Fitting Market Implied Volatility Skew using Three Options - The cross markers are the Black-Scholes implied volatilities from the option prices observed from the market. Especially, the red cross markers are the three market implied stock volatility used to calibration the structural model for both the First Passage Time framework and the Merton framework. The blue line is the calibrated volatility skew using the option pricing model under the First Passage Time framework while the green line is output from the option pricing model under the Merton framework. The market data is the mid-price of call options on Apple's share on date 2009-10-02. The options' time to maturity is 50 days.

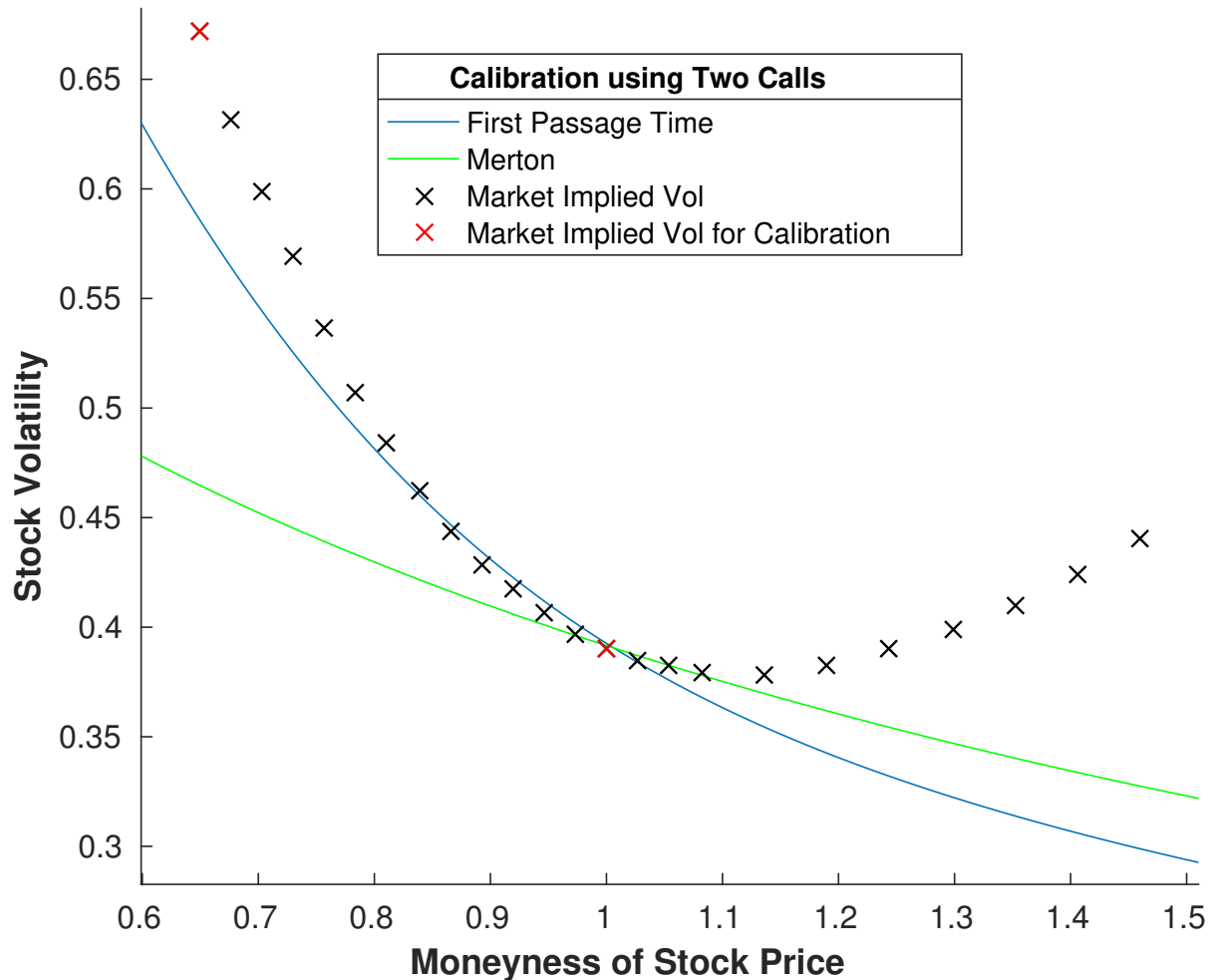


Figure 4.16: Fitting Market Implied Volatility Skew using Two Options - The cross markers are the Black-Scholes implied volatilities from the option prices observed from the market. Especially, the red cross markers are the two market implied stock volatility used to calibration the structural model for both the First Passage Time framework and the Merton framework. The blue line is the calibrated volatility skew using the option pricing model under the First Passage Time framework while the green line is output from the option pricing model under the Merton framework. The market data is the mid-price of call options on Apple's share on date 2009-10-02. The options' time to maturity is 50 days.

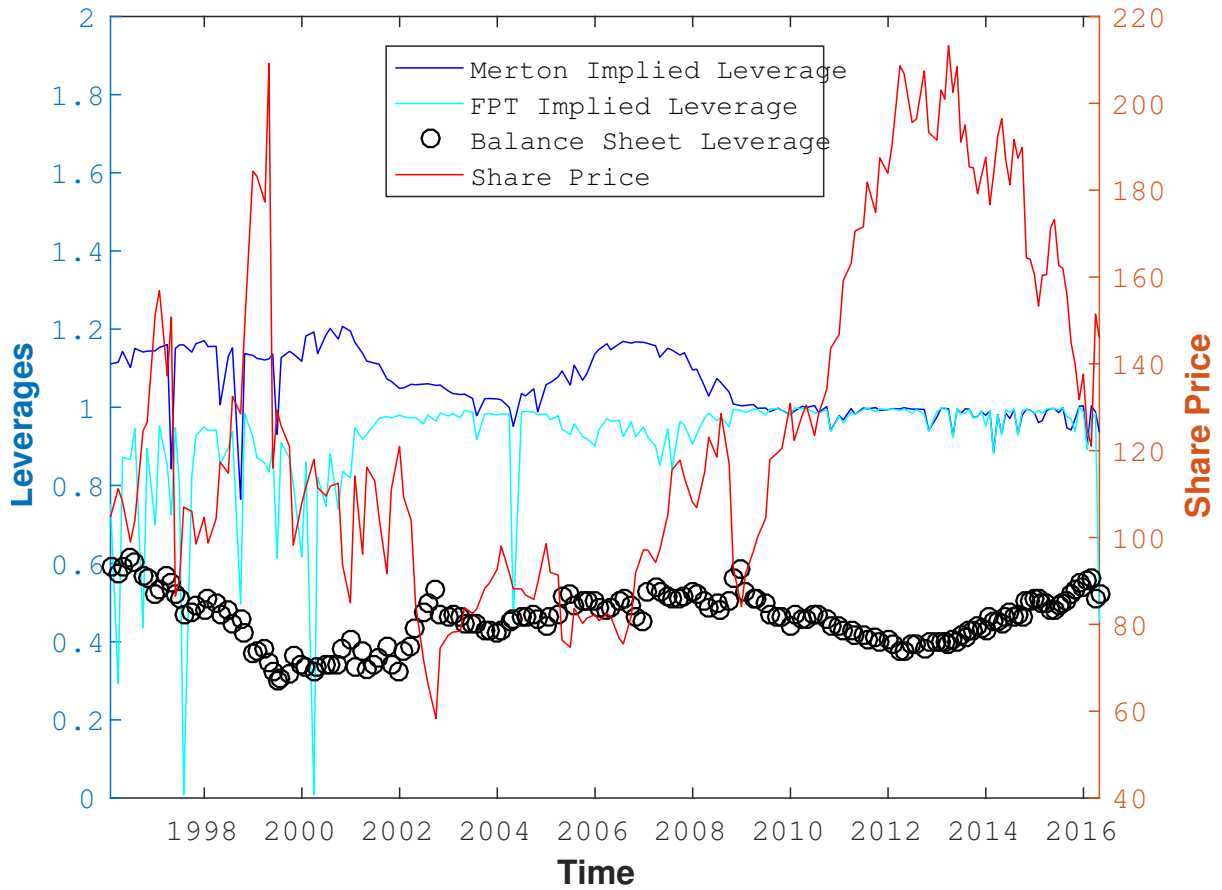


Figure 4.17: Options Implied Leverages, Balance Sheet Leverages and Share Price for IBM - This figure shows option implied leverage L , balance sheet leverage L^B on the left vertical axis and the share price of the equity on the right vertical axis. The graph shows the inverse relationship between the model implied leverages and the share price for both Merton and First Passage Time structural frameworks. Moreover, the option implied leverage under First Passage Time framework is more sensitive to share price change than that of the Merton framework. Leverage L is defined as K/V_t^F . Balance sheet leverage L^B is defined as $K^B/(EQ_t + K^B)$, where K^B is the face value of the zero coupon debt converted from the firm's total liabilities.

Table 4.5: Pricing Errors by Leverage and Option Moneyness - Panels A, B and C report the average pricing errors (in percentage) of Black-Scholes model, option pricing model under Merton framework and option pricing model under the First Passage Time framework. Detailed categorization of option maturity term is describe in Section 4.6.1. The leverage is defined using the market implied leverage by using model under the First Passage Time framework. Low leverage, mid leverage and high leverage are categorized if the market implied leverage $L \in [0, 0.25)$, $L \in [0.25, 0.5)$ and $L \in [0.5, 1)$, respectively.

Panel A: Pricing Errors of Black Scholes Model

Option Moneyness	Low Leverage	Mid Leverage	High Leverage
Very deep in-the-money	0.55	0.87	1.09
Deep in-the-money	0.70	1.18	1.87
In-the-money	0.56	0.91	2.32
At-the-Money	1.05	1.41	4.77
Out-of-the-money	4.35	5.55	20.54
Deep Out-of-the-money	8.80	11.87	37.40
Very deep Out-of-the-money	15.33	23.92	72.73

Panel B: Pricing Errors of Option Pricing Model under Merton Framework

Option Moneyness	Low Leverage	Mid Leverage	High Leverage
Very deep in-the-money	1.06	0.39	0.58
Deep in-the-money	1.70	0.55	1.01
In-the-money	2.60	0.72	1.52
At-the-Money	4.16	1.58	4.25
Out-of-the-money	10.35	5.98	17.07
Deep Out-of-the-money	17.21	13.07	30.82
Very deep Out-of-the-money	31.61	25.13	48.34

Panel C: Pricing Errors of Option Pricing Model under First Passage Time Framework

Option Moneyness	Low Leverage	Mid Leverage	High Leverage
Very deep in-the-money	0.36	0.33	0.50
Deep in-the-money	0.45	0.40	0.88
In-the-money	0.47	0.44	1.33
At-the-Money	1.01	1.16	4.04
Out-of-the-money	4.06	4.96	17.36
Deep Out-of-the-money	8.60	11.60	32.52
Very deep Out-of-the-money	15.54	21.98	49.21

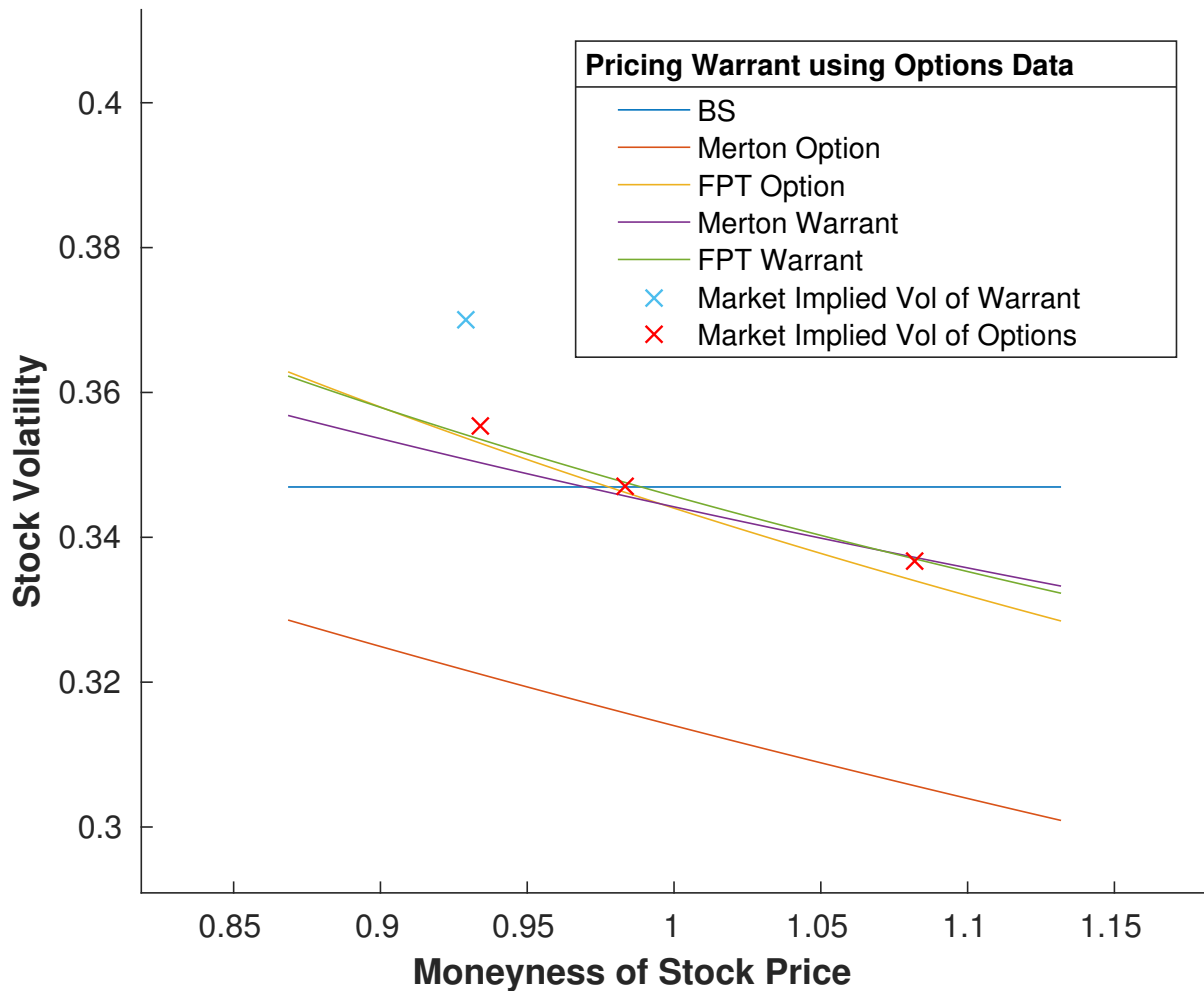


Figure 4.18: Pricing Warrants using Information from Options Data Example 1 - The cross markers in red are the implied volatilities from option prices, which are used to calibrate model parameters. The cross marker in light blue is the Black-Scholes implied volatilities from warrant prices. The implied volatilities skews associated with different models are calculated using the Black-Scholes formula, i.e., warrants are priced at different strikes using the given parameters, and the warrants prices are used to calculate the implied volatilities using the Black-Scholes formula for the call option. The market data is the options and warrant on Agnico Eagle Mines Limited's share on date 2012-12-20. The options' time to maturity is 121 days, while the warrant's time to maturity is 347 days.

Table 4.6: Pricing Errors of Warrants Pricing Model (AEM) - This table reports the average pricing error (in percentage) of warrant pricing models in case study of AEM. BS is the average pricing error of the Black-Scholes Model in pricing warrants. FPT option is the average pricing error of the option pricing model under First Passage Time framework in pricing warrants. Merton Option is the average pricing error of the option pricing model under Merton framework in pricing warrants. FPT warrant is the average pricing error of the warrant pricing model under First Passage Time framework. Merton Warrant is the average pricing error of the warrant pricing model under Merton framework.

Count	BS	FPT Option %	Merton Option %	FPT Warrant %	Merton Warrant %
229	8.79	15.12	31.98	11.07	12.78

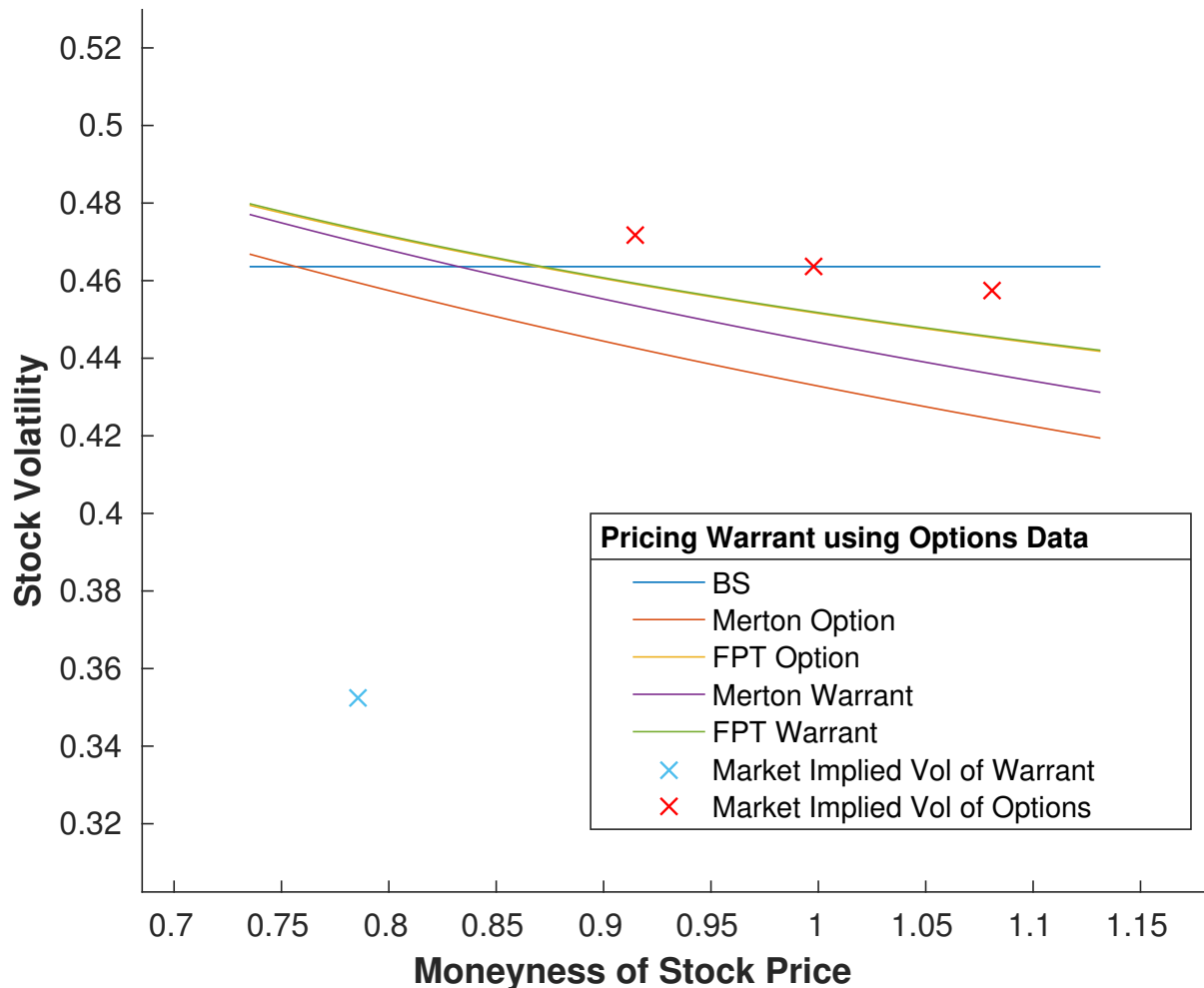


Figure 4.19: Pricing Warrants using Information from Options Data Example 2- The cross markers in red are the implied volatilities from option prices, which are used to calibrate model parameters. The cross marker in light blue is the Black-Scholes implied volatilities from warrant prices. The implied volatilities skews associated with different models are calculated using the Black-Scholes formula, i.e., warrants are priced at different strikes using the given parameters, and the warrants prices are used to calculate the implied volatilities using the Black-Scholes formula for the call option. The market data is the options and warrant on Agnico Eagle Mines Limited's share on date 2009-08-05. The options' time to maturity is 164 days, while the warrant's time to maturity is 1580 days.

Bibliography

- [1] Isabel ABNZANO and Javier F. Navas. Pricing levered warrants with dilution using observable variables. *Quantitative Finance*, 13(8):1199–1209, 2013.
- [2] Fischer Black. How we came up with the option formula. *Journal of Portfolio Management*, 15(2):4–8, 1989.
- [3] Fischer Black and John C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367, 1976.
- [4] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, pages 637–654, 1973.
- [5] Charles Cao, Fan Yu, and Zhaodong Zhong. The information content of option-implied volatility for credit default swap valuation. *Journal of Financial Markets*, 13(3):321 – 343, 2010.
- [6] Peter Carr and Liuren Wu. Stock options and credit default swaps: A joint framework for valuation and estimation. *Journal of Financial Econometrics*, 8(4):409, 2010.
- [7] K.J. Martijn Cremers, Joost Driessen, and Pascal Maenhout. Explaining the level of credit spreads: Option-implied jump risk premia in a firm value model. *The Review of Financial Studies*, 21(5):2209, 2008.
- [8] Michel Crouhy and Dan Galai. Common errors in the valuation of warrants and options on firms with warrants. *Financial Analysts Journal*, 47(5):89–90, 1991.
- [9] Michel Crouhy and Dan Galai. The interaction between the financial and investment decisions of the firm: the case of issuing warrants in a levered firm. *Journal of Banking & Finance*, 18(5):861 – 880, 1994.
- [10] Jin-Chuan Duan. Maximum likelihood estimation using price data of the derivative contract. *Mathematical Finance*, 4(2):155–167, 1994.
- [11] David C. Emanuel. Warrant valuation and exercise strategy. *Journal of Financial Economics*, 12(2):211 – 235, 1983.
- [12] Jan Ericsson and Joel Reneby. Estimating structural bond pricing models*. *The Journal of Business*, 78(2):707–735, 2005.

- [13] Dan Galai and Meir I. Schneller. Pricing of warrants and the value of the firm. *The Journal of Finance*, 33(5):1333–1342, 1978.
- [14] Robert Geske. The valuation of compound options. *Journal of Financial Economics*, 7(1):63 – 81, 1979.
- [15] Robert Geske, Avandhar Subrahmanyam, and Yi Zhou. Capital structure effects on the prices of equity call options. *Journal of Financial Economics*, 121(2):231 – 253, 2016.
- [16] Shmuel Hauser and Beni Lauterbach. The relative performance of five alternative warrant pricing models. *Financial Analysts Journal*, 53(1):55–61, 1997.
- [17] Jing-Zhi Huang and Ming Huang. How much of the corporate-treasury yield spread is due to credit risk? *Review of Asset Pricing Studies*, 2(2):153–202, 2012.
- [18] John Hull, Izzy Nelken, and Alan White. Mertons model, credit risk, and volatility skews. *Journal of Credit Risk*, 1(1), 2004.
- [19] Robert A. Jarrow and Siegfried Trautmann. A reduced-form model for warrant valuation. *Financial Review*, 46(3):413–425, 2011.
- [20] E. Philip Jones, Scott P. Mason, and Eric Rosenfeld. Contingent claims analysis of corporate capital structures: an empirical investigation. *The Journal of Finance*, 39(3):611–625, 1984.
- [21] Beni Lauterbach and Paul Schultz. Pricing warrants: An empirical study of the black-scholes model and its alternatives. *The Journal of Finance*, 45(4):1181–1209, 1990.
- [22] Hayne E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *The Journal of Finance*, 49(4):1213–1252, 1994.
- [23] Hayne E. Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987–1019, 1996.
- [24] Ka Leung Li and Hoi Ying Wong. Structural models of corporate bond pricing with maximum likelihood estimation. *Journal of Empirical Finance*, 15(4):751–777, 2008.
- [25] Francis A. Longstaff. Pricing options with extendible maturities: Analysis and applications. *The Journal of Finance*, 45(3):935–957, 1990.
- [26] Francis A. Longstaff and Eduardo S. Schwartz. A simple approach to valuing risky fixed and floating rate debt. *The Journal of Finance*, 50(3):789–819, 1995.
- [27] Robert C Merton. Theory of rational option pricing. *The Bell Journal of economics and management science*, pages 141–183, 1973.
- [28] Robert C Merton. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, 1974.

- [29] Robert C Merton. *Continuous-time finance*. Blackwell Boston, 1992.
- [30] Franco Modigliani and Merton H Miller. The cost of capital, corporation finance and the theory of investment. *The American Economic Review*, pages 261–297, 1958.
- [31] Ehud I. Ronn and Avinash K. Verma. Pricing risk-adjusted deposit insurance: An option-based model. *The Journal of Finance*, 41(4):871–896, 1986.
- [32] Mark Rubinstein and Eric Reiner. Breaking down the barriers. *Risk*, 4(8):28–35, 1991.
- [33] G.Uwe Schulz and Siegfried Trautmann. Robustness of option-like warrant valuation. *Journal of Banking & Finance*, 18(5):841 – 859, 1994.
- [34] Jakob Sidenius. Warrant pricing: Is dilution a delusion? *Financial Analysts Journal*, 52(5):77–80, 1996.
- [35] Andrey D. Ukhov. Warrant pricing using observable variables. *Journal of Financial Research*, 27(3):329–339, 2004.
- [36] Xinghua Zhou and R. Mark Reesor. Misrepresentation and capital structure: Quantifying the impact on corporate debt value. *Journal of Corporate Finance*, 34:293 – 310, 2015.

Chapter 5

Conclusion

In this thesis, we study the structural model and its applications in damages calculation, option pricing and warrant pricing. In Chapter 2, using a modified Merton framework for valuing corporate securities and a connection between the observable share price and firm value we show the impact of a fraud-induced share price change on the value of the other corporate securities in the capital structure. We show that the impact on debt value depends on firm leverage and debt seniority and not on the warrant dilution factor. Generally, the debt for higher-leverage firms is more sensitive to the misrepresentation impact than lower-leverage firms and junior debt is more affected by fraud than senior debt. The impact on warrant value is determined by warrant moneyness (stock price), with the dilution factor having no effect. Moreover, we demonstrate by a case study the implementation of a damages calculation methodology, which is consistent with the event study approach for equities damages calculation, for warrants and debts. In addition to our main findings, we provide ancillary contributions to the warrant valuation and capital structure literature. We extend Crouhy and Galai's [3] framework to a capital structure that includes both junior and senior debt and discuss the warrant exercise criteria under this structural framework. Additionally, we broaden the connection between observable stock price and its volatility with unobservable firm value and its volatility discussed in Ukhov[7] and Abinzano and Navas[1] to other capital structures not previously considered.

In Chapter 3, we extended the study in Chapter 2 into the First Passage Time framework, which is capable to model firm with more complex debt structures. Using an extended Black and Cox capital structure modelling framework and a connection between the observable share price and firm value we connect the impact of an observable fraud-induced share price change on the debt value. Generally, debt for higher-leverage firms is more sensitive to the fraud size than lower-leverage firms and junior debt is more affected by fraud size than senior debt. This study proposes a methodology to compute damages in securities class actions for investors with debt positions in the fraud-committing company. This work is relevant not only for estimating potential damages, but in the fair allocation of damage awards across holders of the fraudulent firm's securities. The legal requirement that damages due to fraud must be computed net of any hedge or risk limitation transaction underscores the importance of the work presented here. For example, one can hedge a long bond position by shorting the shares of the bond issuer. This methodology allows one to compute fraud-induced share and debt value changes in a consistent manner. In addition to our main findings, we explicitly discuss bankruptcy costs for the First Passage Time model. Furthermore, we are able to reduce a system of two non-linear equations,

used to connect the unobservable firm value and firm value volatility to observable equity value and equity volatility, into one equation. This technique improves the ability to solve the non-linear system.

In Chapter 4 we derive the closed-form formula for European option price under the First Passage Time (FPT) structural framework and extended the calibration method of [6] and [4] to the FPT framework. We show that the model implied volatility skew under FPT framework is much more flexible than that under Merton framework. Following [2], we extended the FPT structural framework to include warrants into a firm's capital structure. Moreover, we derive the European call option price under the warrant extended capital structure model (under both Merton and FPT frameworks). Also, we proposed a calibration method for the warrant extended capital structure model under both Merton and FPT frameworks. As a results, warrants can be priced under a unified framework by using information from European call options. Using a case study with more than thirty thousands options data we show that the option pricing model under FPT framework significantly outperforms the option pricing model under Merton framework. The results of the case study confirm the empirical study in [4] that the option pricing models under structural frameworks outperform the Black-Scholes model, i.e., the equity option price contain insightful information about a firm's capital structure. Moreover, our case study indicates that the FPT framework could better capture the embedded leverage information from options than Merton framework. A case study using AEM's warrants data demonstrates the implementation of our warrant pricing models under both Merton and FPT structural frameworks.

There are many different avenues for future research. Extending the modelling framework study in Chapter 2 and Chapter 3 to more realistic capital structures (e.g., include preferred shares and employee stock options) and to incorporate the callable/convertible features in the bonds are obvious directions to pursue. Using this extended modelling framework, an empirical analysis of the model performance would be required to see how well the model-predicted security value changes match the observed changes. Additionally, one could investigate if misrepresentation is a partial explanation of the credit spread puzzle as analyzed in Huang and Huang [5].

On the options pricing and warrants study, future research will include: performing an empirical study for the option pricing models by using more options data; performing an empirical study for the warrant pricing model; studying the option and warrant pricing model under FPT framework by assuming different firm value dynamics (e.g., CEV process); studying the impact of liquidity in pricing warrants.

Bibliography

- [1] Isabel ABÍNZANO and Javier F. Navas. Pricing levered warrants with dilution using observable variables. *Quantitative Finance*, 13(8):1199–1209, 2013.
- [2] Michel Crouhy and Dan Galai. Common errors in the valuation of warrants and options on firms with warrants. *Financial Analysts Journal*, 47(5):89–90, 1991.
- [3] Michel Crouhy and Dan Galai. The interaction between the financial and investment decisions of the firm: The case of issuing warrants in a levered firm. *Journal of Banking & Finance*, 18(5):861–880, 1994.
- [4] Robert Geske, Avanidhar Subrahmanyam, and Yi Zhou. Capital structure effects on the prices of equity call options. *Journal of Financial Economics*, 121(2):231 – 253, 2016.
- [5] Jing-Zhi Huang and Ming Huang. How much of the corporate-treasury yield spread is due to credit risk? *Review of Asset Pricing Studies*, 2(2):153–202, 2012.
- [6] John Hull, Izzy Nelken, and Alan White. Mertons model, credit risk, and volatility skews. *Journal of Credit Risk*, 1(1), 2004.
- [7] Andrey D Ukhov. Warrant pricing using observable variables. *Journal of Financial Research*, 27(3):329–339, 2004.

Appendix A

Appendix for Chapter 2

A.1 Some Pricing Algorithms in Section 2.2

A.1.1 Pricing Stocks, Warrants and Debt in Crouhy and Galai's Model of Section 4.4

Input: V_t^F , T , T_w , t , σ^F , K , F , M , N , and r .

Output: D_t , W_t , and S_t .

- Find \bar{V}^F by solving

$$x - D(x + MK, F, T - T_w, \sigma^F, r) = NK, \quad (\text{A.1})$$

for x , where $D(V, F, \tau, \sigma^F, r)$ is the debt value function in Merton's model, given by Equation (2.9).

- Compute \bar{Z} using Equation (2.22).
- Compute S_t , D_t , and W_t using Equations (2.23), (2.24), and (4.32). There is no analytical solution for these integrations. We need to calculate them numerically. Note that $V_{T_w}^{F*}$ is given by Equation (4.27), where $V_t^{F*} = V_t^F$.

A.1.2 Pricing Stocks, Warrants, Junior Debt, and Senior Debt in Section 2.2.5

Input: V_t^F , T , T_w , t , σ^F , K , F^J , F^S , M , N , and r .

Output: D_t^J , D_t^S , W_t , and S_t .

- Find \bar{V}^F by solving

$$x - D^J(x + MK, F^J, F^S, T - T_w, \sigma^F, r) - D^S(x + MK, F^J, F^S, T - T_w, \sigma^F, r) = NK, \quad (\text{A.2})$$

for x , where $D^J(V, F^J, F^S, \tau, \sigma^F, r)$ and $D^S(V, F^J, F^S, \tau, \sigma^F, r)$ are given by Equations (B.24) and (B.23).

- Compute \bar{Z} using Equation (2.22) by using the \bar{V}^F above.
- Compute S_t , D_t^J , D_t^S , and W_t using Equations (2.28), (2.30), (2.29), and (2.31). There is no analytical solution for these integrations. We need to calculate them numerically. Note that $V_{T_w}^{F*}$ is given by Equation (4.27), where $V_t^{F*} = V_t^F$.

A.2 Connection Between (V_t^F, σ^F) and (S_t, σ^S)

A.2.1 Warrants and Common Shares Model

For this capital structure, the firm value is the sum of the values of warrants and common shares. We have

$$V_t^F = NS_t + MW_t. \quad (\text{A.3})$$

We can compute the partial derivative $\frac{\partial S_t}{\partial V_t^F}$ using Equation (2.19). So we have

$$\begin{aligned} \frac{\partial S_t}{\partial V_t^F} &= \frac{1}{N} - \frac{M}{N} \frac{\partial W_t}{\partial V_t^F} \\ &= \frac{1}{N} - \frac{M}{N(M+N)} N(u_1), \end{aligned} \quad (\text{A.4})$$

where u_1 is given by

$$u_1 = (\log(V_t^F / (NK)) + (r + \frac{1}{2}\sigma^{F^2})(T_w - t)) / (\sigma^F \sqrt{T_w - t}). \quad (\text{A.5})$$

Coupled with Equation (A.3), the link between the pairs (V_t^F, σ^F) and (S_t, σ^S) is established, namely,

$$\begin{cases} \sigma^S &= \sigma^F \frac{M+N-MN(u_1)}{N(M+N)} \frac{V_t^F}{S_t} \\ V_t^F &= NS_t + \frac{M}{M+N} C(V_t^F, NK, T_w - t). \end{cases} \quad (\text{A.6})$$

Given the value of (S_t, σ^S) , Ukhov[2] shows that this system has a solution $(\bar{V}_t^F, \bar{\sigma}^F)$.

A.2.2 Debt, Warrants, and Common Shares Model

To establish the connection between the pairs (V_t^F, σ^F) and (S_t, σ^S) in this model, we follow the same routine as in Ukhov[2] and Abinzano and Navas[1]. First, we find the relation between S_t and (V_t^F, σ^F) from the pricing formula in Equation (2.23). Also, it can be shown by Ito's formula that

$$\sigma^S = \sigma^F \frac{\partial S_t}{\partial V_t^F} \frac{V_t^F}{S_t}. \quad (\text{A.7})$$

Combining these two equations, we have constructed the system of non-linear equations

$$\begin{cases} \sigma^S &= \sigma^F \frac{\partial S_t}{\partial V_t^F} \frac{V_t^F}{S_t} \\ S_t &= e^{-r(T_w-t)} \frac{1}{\sqrt{2\pi(T_w-t)}} \\ &\quad \times \left\{ \int_{\bar{Z}}^{+\infty} \frac{C(V_{T_w}^{F*} + MK, F, T - T_w)}{M+N} \exp\left\{\frac{-x^2}{2(T_w-t)}\right\} dx \right. \\ &\quad \left. + \int_{-\infty}^{\bar{Z}} \frac{C(V_{T_w}^{F*}, F, T - T_w)}{N} \exp\left\{\frac{-x^2}{2(T_w-t)}\right\} dx \right\}, \end{cases} \quad (\text{A.8})$$

where \bar{Z} is given by Equation (2.22), $\partial S_t / \partial V_t^F$ is given by

$$\begin{aligned} \frac{\partial S_t}{\partial V_t^F} = & \frac{e^{-r(T_W-t)}}{\sqrt{2\pi(T_W-t)}} \left\{ \int_{\bar{Z}}^{+\infty} \frac{N(l_1)}{M+N} \exp\left\{\left(r - \frac{1}{2}\sigma^{F^2}\right)(T_W-t) + \sigma^F \sqrt{T_W-t}x\right. \right. \\ & \left. \left. - \frac{x^2}{2(T_W-t)}\right\} dx + \frac{C(\bar{V}^F + MK, F, T - T_W)}{M+N} \exp\left\{\frac{-\bar{Z}^2}{2(T_W-t)}\right\} \frac{1}{V_t^F \sigma^F \sqrt{T_W-t}} \right. \\ & + \int_{-\infty}^{\bar{Z}} \frac{N(m_1)}{N} \exp\left\{\left(r - \frac{1}{2}\sigma^{F^2}\right)(T_W-t) + \sigma^F \sqrt{T_W-t}x - \frac{x^2}{2(T_W-t)}\right\} dx \\ & \left. - \frac{C(\bar{V}^F, F, T - T_W)}{N} \exp\left\{\frac{-\bar{Z}^2}{2(T_W-t)}\right\} \frac{1}{V_t^F \sigma^F \sqrt{T_W-t}} \right\}, \end{aligned} \quad (\text{A.9})$$

$$l_1 = \frac{\log\left(\frac{V_t^F \exp\{(r-0.5\sigma^{F^2})(T_W-t) + \sigma^F \sqrt{T_W-t}x\} + MK}{F}\right) + (r + \frac{1}{2}\sigma^{F^2})(T - T_W)}{\sigma^F \sqrt{T - T_W}}, \text{ and} \quad (\text{A.10})$$

$$m_1 = \frac{\log\left(\frac{V_t^F \exp\{(r-0.5\sigma^{F^2})(T_W-t) + \sigma^F \sqrt{T_W-t}x\}}{F}\right) + (r + \frac{1}{2}\sigma^{F^2})(T - T_W)}{\sigma^F \sqrt{T - T_W}}. \quad (\text{A.11})$$

A.2.3 Junior, Senior Debt, Warrants, and Common Shares Model

As mentioned before, we can view the junior and senior debt as bundled debt with face value $F^J + F^S$. So the system of non-linear equations is similar to that in the previous section. We have

$$\begin{cases} \sigma^S &= \sigma^F \frac{\partial S_t}{\partial V_t^F} \frac{V_t^F}{S_t} \\ S_t &= e^{-r(T_W-t)} \frac{1}{\sqrt{2\pi(T_W-t)}} \\ &\quad \times \left\{ \int_{\bar{Z}}^{+\infty} \frac{C(V_{TW}^{F^S} + MK, F^J + F^S, T - T_W)}{M+N} \exp\left\{\frac{-x^2}{2(T_W-t)}\right\} dx \right. \\ &\quad \left. + \int_{-\infty}^{\bar{Z}} \frac{C(V_{TW}^{F^S}, F^J + F^S, T - T_W)}{N} \exp\left\{\frac{-x^2}{2(T_W-t)}\right\} dx \right\}, \end{cases} \quad (\text{A.12})$$

where \bar{Z} is given in Equation (2.22) with \bar{V}^F defined in Section 2.2.5. Again, we need to compute $\frac{\partial S_t}{\partial V_t^F}$ numerically.

A.3 Hedging

The partial derivative $\partial W_t / \partial V_t^F$ is given by

$$\begin{aligned} \frac{\partial W_t}{\partial V_t^F} = & \frac{e^{-r(T_W-t)}}{\sqrt{2\pi(T_W-t)}} \left\{ \int_{\bar{Z}}^{+\infty} \frac{N(l_1)}{M+N} \exp\left\{\left(r - \frac{1}{2}\sigma^{F^2}\right)(T_W-t) + \sigma^F \sqrt{T_W-t}x\right. \right. \\ & \left. \left. - \frac{x^2}{2(T_W-t)}\right\} dx + \left(\frac{C(\bar{V}^F + MK, F, T - T_W)}{M+N} - K \right) \right. \\ & \left. \times \exp\left\{\frac{-\bar{Z}^2}{2(T_W-t)}\right\} \frac{1}{V_t^F \sigma^F \sqrt{T_W-t}} \right\} \end{aligned} \quad (\text{A.13})$$

Bibliography

- [1] Isabel ABÍNZANO and Javier F. Navas. Pricing levered warrants with dilution using observable variables. *Quantitative Finance*, 13(8):1199–1209, 2013.
- [2] Andrey D Ukhov. Warrant pricing using observable variables. *Journal of Financial Research*, 27(3):329–339, 2004.

Appendix B

Appendix for Chapter 3

B.1 Formula of European Down-and-Out Call Options

Under this modelling framework, the value of a European down-and-out call option written on V^F with barrier K_B and strike price K_S ($K_B \leq K_S$) is

$$C_{DO} = V_t^F \Phi(x_1) - K_S e^{-r(T-t)} \Phi(x_1 - \sigma^F \sqrt{T-t}) - V_t^F (K_B/V_t^F)^{2\lambda} \Phi(y_1) + K_S e^{-r(T-t)} (K_B/V_t^F)^{2\lambda-2} \Phi(y_1 - \sigma^F \sqrt{T-t}), \quad (\text{B.1})$$

where

$$\lambda = (r + \sigma^{F^2}/2)/\sigma^{F^2}, \quad (\text{B.2})$$

$$x_1 = (\log(V_t^F/K_S) + (r + \sigma^{F^2}/2)(T-t))/(\sigma^F \sqrt{T-t}), \quad (\text{B.3})$$

and

$$y_1 = (\log(K_B^2/(V_t^F K_S)) + (r + \sigma^{F^2}/2)(T-t))/(\sigma^F \sqrt{T-t}). \quad (\text{B.4})$$

If $K_B \geq K_S$, the value is

$$C_{DO} = V_t^F \Phi(x_2) - K_S e^{-r(T-t)} \Phi(x_2 - \sigma^F \sqrt{T-t}) - V_t^F (K_B/V_t^F)^{2\lambda} \Phi(y_2) + K_S e^{-r(T-t)} (K_B/V_t^F)^{2\lambda-2} \Phi(y_2 - \sigma^F \sqrt{T-t}), \quad (\text{B.5})$$

where

$$x_2 = (\log(V_t^F/K_B) + (r + \sigma^{F^2}/2)(T-t))/(\sigma^F \sqrt{T-t}), \quad (\text{B.6})$$

and

$$y_2 = (\log(K_B/V_t^F) + (r + \sigma^{F^2}/2)(T-t))/(\sigma^F \sqrt{T-t}). \quad (\text{B.7})$$

B.2 Proof of Proposition 3.3.1

Proof We only show the proof of the case where $K_B \leq K_S$ and EQ_t is given by equation (B.1). When $K_B \geq K_S$, the proof is similar to the case where $K_B \leq K_S$.

To show that EQ_t is a one-to-one function of V_t^F , we show that the following statements are true:

- 1 EQ_t approaches zero as the firm value V_t^F approaches the default boundary K_B .
- 2 EQ_t approaches infinity as the firm value approaches infinity.
- 3 EQ_t is an increasing function of V_t^F .

To simplify notation, we let $\bar{x}_1 = x_1 - \sigma^F \sqrt{T-t}$ and $\bar{y}_1 = y_1 - \sigma^F \sqrt{T-t}$.

To show statement one is true, note that as V_t^F approaches K_B , K_B/V_t^F and V_t^F/K_B approach one, and the differences between x_1 and y_1 , and \bar{x}_1 and \bar{y}_1 approach zero, where x_1 and y_1 are given by equations (B.3) and (B.4), respectively. By equation (B.1), the limiting value of equity is

$$\lim_{V_t^F \rightarrow K_B} EQ_t = K_B \Phi(x_1) - K_S e^{-r(T-t)} \Phi(\bar{x}_1) - K_B 1^{2\lambda} \Phi(x_1) + K_S e^{-r(T-t)} 1^{2\lambda-2} \Phi(\bar{x}_1) = 0.$$

To show that the second statement is true, note that as V_t^F approaches infinity, x_1 and \bar{x}_1 approach infinity, and y_1 and \bar{y}_1 approach negative infinity. So $\Phi(x_1)$ and $\Phi(\bar{x}_1)$ approach one, and $\Phi(y_1)$ and $\Phi(\bar{y}_1)$ approach zero. From equation (B.1) the equity value is

$$\begin{aligned} EQ_t &= V_t^F \Phi(x_1) - K_S e^{-r(T-t)} \Phi(\bar{x}_1) - V_t^F (K_B/V_t^F)^{2\lambda} \Phi(y_1) + K_S e^{-r(T-t)} (K_B/V_t^F)^{2\lambda-2} \Phi(\bar{y}_1) \\ &\geq V_t^F \Phi(x_1) - K_S e^{-r(T-t)} \Phi(\bar{x}_1) - K_B^{2\lambda} V_t^{F^{1-2\lambda}} \Phi(y_1). \end{aligned}$$

By the definition of λ (see equation(4.7)) we have

$$1 - 2\lambda = -\frac{2r}{\sigma^F{}^2}, \quad (\text{B.8})$$

which is negative, and so when V_t^F approaches infinity, $V_t^{F^{1-2\lambda}}$ approaches zero. Hence,

$$\begin{aligned} \lim_{V_t^F \rightarrow \infty} EQ_t &\geq \lim_{V_t^F \rightarrow \infty} [V_t^F \Phi(x_1) - K_S e^{-r(T-t)} \Phi(\bar{x}_1) - K_B^{2\lambda} V_t^{F^{1-2\lambda}} \Phi(y_1)] \\ &= \infty - K_S e^{-r(T-t)} - 0 = \infty. \end{aligned}$$

For statement three, we show that EQ_t is an increasing function of V_t^F by showing its derivative w.r.t. V_t^F is positive. Taking the partial derivative we have

$$\begin{aligned} \frac{\partial EQ_t}{\partial V_t^F} &= (V_t^F \Phi(x_1) - K_S e^{-r(T-t)} \Phi(\bar{x}_1))' - (V_t^F (K_B/V_t^F)^{2\lambda} \Phi(y_1) - K_S e^{-r(T-t)} (K_B/V_t^F)^{2\lambda-2} \Phi(\bar{y}_1))' \\ &= (V_t^F \Phi(x_1) - K_S e^{-r(T-t)} \Phi(\bar{x}_1))' - ((V_t^F (K_B/V_t^F)^{2\lambda})' \Phi(y_1) + V_t^F (K_B/V_t^F)^{2\lambda} \Phi'(y_1) \\ &\quad - K_S e^{-r(T-t)} ((K_B/V_t^F)^{2\lambda-2})' \Phi(\bar{y}_1) - K_S e^{-r(T-t)} (K_B/V_t^F)^{2\lambda-2} \Phi'(\bar{y}_1)), \end{aligned} \quad (\text{B.9})$$

where $(\cdot)'$ denotes differentiation with respect to V_t^F . The first term in equation (B.9) is the delta of the European call option written on firm value and struck at K_S in the Black-Scholes [2] model, and is equal to $\Phi(x_1)$. It can be shown through straightforward and tedious calculation that

$$(V_t^F (K_B/V_t^F)^{2\lambda} \Phi'(y_1)) / (K_S e^{-r(T-t)} (K_B/V_t^F)^{2\lambda-2} \Phi'(\bar{y}_1)) = 1.$$

Noting that $V_t^F(K_B/V_t^F)^{2\lambda}\Phi'(y_1) = K_S e^{-r(T-t)}(K_B/V_t^F)^{2\lambda-2}\Phi'(\bar{y}_1)$ equation (B.9) becomes

$$\begin{aligned} \frac{\partial EQ_t}{\partial V_t^F} &= \Phi(x_1) - ((V_t^F(K_B/V_t^F)^{2\lambda})'\Phi(y_1) - K_S e^{-r(T-t)}((K_B/V_t^F)^{2\lambda-2})'\Phi(\bar{y}_1)) \\ &= \Phi(x_1) - ((1-2\lambda)(K_B/V_t^F)^{2\lambda}\Phi(y_1) - (2-2\lambda)e^{-r(T-t)}(K_B/V_t^F)^{2\lambda-1}\Phi(\bar{y}_1)(K_S/K_B)) \\ &= \Phi(x_1) + (2r/\sigma^{F^2})(K_B/V_t^F)^{2\lambda}\Phi(y_1) + (1-2r/\sigma^{F^2})(K_B/V_t^F)^{2\lambda-1}e^{-r(T-t)}\Phi(\bar{y}_1)(K_S/K_B) \end{aligned} \quad (\text{B.10})$$

If $1 - 2r/\sigma^{F^2} \geq 0$, the derivative in equation (B.10) positive, because all the terms are positive. So we only need to show that equation (B.10) is positive when $1 - 2r/\sigma^{F^2} < 0$. Since $V_t^F \geq K_B$ and $K_S \geq K_B$, we have $x_1 \geq y_1$ by $(V_t^F/K_S)/(K_B^2/(V_t^F K_S)) \geq 1$. Hence $\Phi(x_1) \geq \Phi(y_1) \geq \Phi(\bar{y}_1)$. Combining this and the assumption of $1 - 2r/\sigma^{F^2} < 0$ and $K_S/K_B \geq 1$, we have

$$\begin{aligned} \frac{\partial EQ_t}{\partial V_t^F} &\geq \Phi(\bar{y}_1) + (2r/\sigma^{F^2})(K_B/V_t^F)^{2\lambda}\Phi(\bar{y}_1) + (1-2r/\sigma^{F^2})(K_B/V_t^F)^{2\lambda-1}\Phi(\bar{y}_1) \\ &= \Phi(\bar{y}_1)(K_B/V_t^F)^{2\lambda}((V_t^F/K_B)^{2\lambda} + 2r/\sigma^{F^2} + (1-2r/\sigma^{F^2})(V_t^F/K_B)) \end{aligned} \quad (\text{B.11})$$

Expanding $(V_t^F/K_B)^{2\lambda}$ around the point 1 we get

$$(V_t^F/K_B)^{2\lambda} = 1^{2\lambda} + 2\lambda 1^{2\lambda-1}(V_t^F/K_B - 1) + R_2,$$

where R_2 is the reminder term given by

$$R_2 = \frac{2\lambda(2\lambda-1)\xi^{2\lambda-2}}{2!}(V_t^F/K_B - 1)^2,$$

with $\xi \in [1, V_t^F/K_B]$. Since $R_2 \geq 0$, we have

$$(V_t^F/K_B)^{2\lambda} \geq 1 + 2\lambda(V_t^F/K_B - 1),$$

and hence

$$\begin{aligned} \frac{\partial EQ_t}{\partial V_t^F} &\geq \Phi(\bar{y}_1)(K_B/V_t^F)^{2\lambda}(1 + 2\lambda(V_t^F/K_B - 1) + 2r/\sigma^{F^2} + (1-2r/\sigma^{F^2})(V_t^F/K_B)) \\ &= \Phi(\bar{y}_1)(K_B/V_t^F)^{2\lambda}(1 + \frac{(2r + \sigma^{F^2})(V_t^F - K_B)}{\sigma^{F^2}K_B} + \frac{2rK_B}{\sigma^{F^2}K_B} + \frac{(\sigma^{F^2} - 2r)V_t^F}{\sigma^{F^2}K_B}) \\ &\geq \Phi(\bar{y}_1)(K_B/V_t^F)^{2\lambda}(1 + \frac{2r(V_t^F - K_B) + 2rK_B + (\sigma^{F^2} - 2r)V_t^F}{\sigma^{F^2}K_B}) \\ &= \Phi(\bar{y}_1)(K_B/V_t^F)^{2\lambda}(1 + V_t^F/K_B) > 0. \end{aligned}$$

B.3 Proof of Theorem 3.3.2

Proof Given EQ_t and σ^E , if $\bar{\sigma}^F$ satisfies equation (3.26), the firm value $\bar{V}_t^F = v_{EQ_t}(\bar{\sigma}^F)$ and $\bar{\sigma}^F$ satisfy equation (3.23) by the definition of function v_{EQ_t} . Since equation (3.26) is derived by combining (3.23) and (3.24), $\bar{\sigma}^F$ and \bar{V}_t^F must satisfy equation (3.24).

If $\bar{\sigma}^F$ and \bar{V}_t^F satisfy the non-linear system (3.23) and (3.24), $\bar{V}_t^F = v_{EQ_t}(\bar{\sigma}^F)$ by the definition of function v_{EQ_t} . So $\bar{\sigma}^F$ satisfies equation (3.26).

B.4 The Maximum Likelihood Method for FPT Model

In this section we provide the details of the MLE method introduced in [3], [4], [5] and [6]. Given the time series of equity value $EQ^{TS} \equiv \{EQ_i^{ts} : i = 1, 2, \dots, n\}$, firm volatility σ^F can be estimated by MLE, and hence the firm value V_t^F can be computed using the function $V_t^F = v_{EQ_t}(\sigma^F)$ which is defined in Theorem 3.3.2.

Let $f(\cdot)$ be the conditional density of EQ_i^{ts} given EQ_{i-1}^{ts} , the log-likelihood function for vector EQ^{TS} is

$$L_{EQ}(EQ^{TS}; \sigma^F, \lambda^v) = \sum_{i=2}^n \log f(EQ_i^{ts} | EQ_{i-1}^{ts}; \sigma^F, \lambda^v), \quad (\text{B.12})$$

where λ^v is the market price of risk given in the firm dynamic (see equation (4.1)). From the dynamic of V_t^F , it is easy to show that logarithm of V_t^F is normally distributed, and its conditional density function $g(\cdot)$ is given by

$$g(\log V_i^F | \log V_{i-1}^F; \sigma^F, \lambda^v) = \frac{1}{\sqrt{2\pi s_i^2}} \exp\left(-\frac{(\log V_i^F - m_i)^2}{2s_i^2}\right), \quad (\text{B.13})$$

where

$$m_i = \log V_{i-1}^F + (r + \lambda^v \sigma^F - 0.5\sigma^{F^2})\Delta t, \quad (\text{B.14})$$

$$s_i = \sigma^F \sqrt{\Delta t}, \quad (\text{B.15})$$

and $\Delta t = t_i - t_{i-1}$. By using the fact that V_t^F is a one-to-one and hence invertible function of EQ_t (see Proposition 3.3.1), the conditional density $f(\cdot)$ can be written as

$$f(EQ_i^{ts} | EQ_{i-1}^{ts}; \sigma^F, \lambda^v) = g(\log V_i^F | \log V_{i-1}^F; \sigma^F, \lambda^v) \Big|_{V_i^F = v_{EQ_i^{ts}}(\sigma^F)} \times \left(\frac{\partial EQ_i}{\partial \log V_i^F} \Big|_{V_i^F = v_{EQ_i^{ts}}(\sigma^F)} \right)^{-1}. \quad (\text{B.16})$$

Substituting equations (B.10) and (B.16) into the log-likelihood function (B.12) gives

$$\begin{aligned} L_{EQ}(EQ^{TS}; \sigma^F, \lambda^v) &= \sum_{i=2}^n \left(\log g(\log V_i^F | \log V_{i-1}^F; \sigma^F, \lambda^v) \Big|_{V_i^F = v_{EQ_i^{ts}}(\sigma^F)} - \log \frac{\partial EQ_i}{\partial \log V_i^F} \Big|_{V_i^F = v_{EQ_i^{ts}}(\sigma^F)} \right) \\ &= \sum_{i=2}^n \left(-\frac{1}{2} \log(2\pi s_i^2) - \frac{(\log V_i^F - m_i)^2}{2s_i^2} \right. \\ &\quad \left. - \log(V_i^F (\Phi(d_1) + (2r/\sigma^{F^2})(K/V_i^F)^{2\lambda} \Phi(d_3)) \right. \\ &\quad \left. + (1 - 2r/\sigma^{F^2})(K/V_i^F)^{2\lambda-1} e^{-r(T-t_i)} \Phi(d_4))) \right) \Big|_{V_i^F = v_{EQ_i^{ts}}(\sigma^F)}. \end{aligned} \quad (\text{B.17})$$

Taking into account the survivorship issue [5], the log-likelihood function for the First Passage Time model is

$$\begin{aligned} L_{EQ}^{FPT}(EQ^{TS}; \sigma^F, \lambda^v) &= L_{EQ}(EQ^{TS}; \sigma^F, \lambda^v) + \log \left(\Pr(\mathcal{D}_n | v_{EQ_0^{ts}}(\sigma^F), v_{EQ_1^{ts}}(\sigma^F), \dots, v_{EQ_n^{ts}}(\sigma^F)) \right) \\ &\quad - \log(\Pr(\mathcal{D}_n)), \end{aligned} \quad (\text{B.18})$$

where \mathcal{D}_i is the event that the firm does not default up to time t_i and $\Pr(\cdot)$ is the probability under the physical measure. The expressions of second term and third term of (B.18) is given by¹

$$\begin{aligned} & \log \left(\Pr(\mathcal{D}_n | v_{EQ_0^s}(\sigma^F), v_{EQ_1^s}(\sigma^F), \dots, v_{EQ_n^s}(\sigma^F)) \right) \\ &= \sum_{i=2}^n \log \left(1 - \exp \left(\frac{2}{\sigma^{F^2} \Delta t} \log \left(\frac{v_{EQ_i^s}(\sigma^F)}{K} \right) \log \left(\frac{K}{v_{EQ_{i-1}^s}(\sigma^F)} \right) \right) \right), \end{aligned} \quad (\text{B.19})$$

and

$$\begin{aligned} \log(\Pr(\mathcal{D}_n)) &= \log \left(\Phi \left(\frac{(r + \lambda^v \sigma^F - \frac{\sigma^{F^2}}{2})n\Delta t - \log(\frac{K}{v_{EQ_0^s}(\sigma^F)})}{\sqrt{n\Delta t}\sigma^F} \right) \right. \\ &\quad \left. - \exp \left(\frac{2(r + \lambda^v \sigma^F - \frac{\sigma^{F^2}}{2})}{\sigma^{F^2}} \log \left(\frac{K}{v_{EQ_0^s}(\sigma^F)} \right) \right) \Phi \left(\frac{(r + \lambda^v \sigma^F - \frac{\sigma^{F^2}}{2})n\Delta t + \log(\frac{K}{v_{EQ_0^s}(\sigma^F)})}{\sqrt{n\Delta t}\sigma^F} \right) \right). \end{aligned} \quad (\text{B.20})$$

Maximizing the log-likelihood function (B.18) gives the estimators $\hat{\sigma}^F$ and $\hat{\lambda}^v$ from which we get the estimated firm value $\hat{V}_t^F = v_{EQ_t}(\hat{\sigma}^F)$.

B.5 The Subordinated Debt Model

In this section we briefly introduce the Subordinated Debt model under Merton's framework.² Subordinated debt has a lower priority than senior debt at the time of liquidation and the absolute priority rule holds. Let D^S and D^J denote the senior and subordinated debt values, respectively. Assume that both types of debt are zero-coupon and have the same maturity date T , which is also the liquidation date. The time- T payoffs for senior debt, subordinated debt, and equity on date T are given in Table B.1. F^S and F^J are the face values of senior and junior

Table B.1: Time- T Payoff

	Pay-off at maturity date T
<i>Equity</i>	$\max\{V_T^F - F^S - F^J, 0\}$
<i>JuniorDebt</i>	$\min\{\max\{V_T^F - F^S, 0\}, F^J\}$
<i>SeniorDebt</i>	$\min\{V_T^F, F^S\}$

debt, respectively. We can evaluate the time- t prices based on the payoffs given in Table B.1. The equity payoff at time T is the same as a European call option on V^F with strike $F^S + F^J$. Thus the time- t equity value can be written as

$$EQ_t = C(V_t^F, F^J + F^S, \sigma^F, r, T - t), \quad (\text{B.21})$$

¹Derivation of these equations can be found in [5]

²Black and Cox [1] price subordinated bonds using a very similar modelling set up.

where $C(\cdot)$ is the Black-Scholes formula for a European call option. It is easily shown that the payoff of the sum of junior debt and equity is the same as a European call on V^F with strike price F^S , i.e.,

$$D_t^J + EQ_t = C(V_t^F, F^S, \sigma^F, r, T - t). \quad (\text{B.22})$$

Hence the value of senior debt D_t^S is

$$\begin{aligned} D_t^S &= V_t^F - (EQ_t + D_t^J) \\ &= V_t^F - C(V_t^F, F^S, \sigma^F, r, T - t). \end{aligned} \quad (\text{B.23})$$

The junior debt position can also be expressed as a call bull spread, which has value

$$\begin{aligned} D_t^J &= V_t^F - D_t^S - EQ_t \\ &= C(V_t^F, F^S, \sigma^F, r, T - t) - C(V_t^F, F^S + F^J, \sigma^F, r, T - t). \end{aligned} \quad (\text{B.24})$$

In the Merton model, the equity value EQ_t and equity volatility σ^E are connected to the firm value V_t^F and firm value volatility σ^F through the following non-linear system:

$$\sigma^E = \sigma^F \frac{\Phi(k_1) V_t^F}{EQ_t}, \quad \text{and} \quad (\text{B.25})$$

$$EQ_t = C(V_t^F, F, \sigma^F, r, T - t), \quad (\text{B.26})$$

where

$$k_1 = (\log(V_t^F / (F^J + F^S)) + (r + \sigma^{F^2}/2)(T - t)) / (\sigma^F \sqrt{T - t}). \quad (\text{B.27})$$

As with the FPT model, this system of equations can be simplified to a single equation. Note that one can also use the MLE method for this model.

Bibliography

- [1] Fischer Black and John C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367, 1976.
- [2] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, pages 637–654, 1973.
- [3] Jin-Chuan Duan. Maximum likelihood estimation using price data of the derivative contract. *Mathematical Finance*, 4(2):155–167, 1994.
- [4] Jin-Chuan Duan. Correction: Maximum likelihood estimation using price data of the derivative contract (mathematical finance 1994, 4/2, 155167). *Mathematical Finance*, 10(4):461–462, 2000.
- [5] Jin-Chuan Duan, Genevieve Gauthier, and Jean-Guy Simonato. On the equivalence of the KMV and maximum likelihood methods for structural credit risk models. Working Paper, 2005.
- [6] Jan Ericsson and Joel Reneby. Estimating structural bond pricing models*. *The Journal of Business*, 78(2):707–735, 2005.

Appendix C

Appendix for Chapter 4

C.1 First Passage Time Model with Bankruptcy Cost

For completeness we discuss the debt valuation and expected bankruptcy cost of the first passage time framework in this section. Please refer to [6] for detailed discussion.

Let w be the loss given default. The payoff of the risky debt with face value 1 and maturity T is

$$1 - wI_{\tau \leq T}, \quad (\text{C.1})$$

where τ is the first passage time of the firm value V^F to boundary K , I_A is an indicator function of event A . The time- t price of a T -maturity risky debt with face value one, $P(V_t^F, \sigma^F, r, w, K, t, T)$, is

$$P(V_t^F, \sigma^F, r, w, K, t, T) = e^{-r(T-t)} E_Q[1 - wI_{\tau \leq T}] = e^{-r(T-t)} (1 - wQ(\tau \leq T)), \quad (\text{C.2})$$

where $Q(\tau \leq T)$ is the probability of the event $[\tau \leq T]$ under the risk-neutral measure. The time- t conditional distribution of the first passage time, τ , is

$$\begin{aligned} Q(\tau \leq T) = & \Phi\left(\frac{-\log(V^F/K) - r(T-t) + 0.5\sigma^{F^2}(T-t)}{\sigma^F \sqrt{T-t}}\right) \\ & + \exp\left(\frac{-2\log(V^F/K)(r - 0.5\sigma^{F^2})}{\sigma^{F^2}}\right) \Phi\left(\frac{-\log(V^F/K) + r(T-t) - 0.5\sigma^{F^2}(T-t)}{\sigma^F \sqrt{T-t}}\right), \end{aligned} \quad (\text{C.3})$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.¹ The time- t value of a zero coupon bond with face value F is $F \times P(V_t^F, \sigma^F, r, w, K, t, T)$. Let D_t be the present value of the firm's total liabilities, which is calculated as the sum of all the bond prices. \bar{T} denotes the weighted average maturity date of all debts. Let $PV(D)$ be the present value of the debts' face value discounted at the risk free rate.

When the firm value V^F reaches the default boundary K at time $\tau < \bar{T}$, the equity value becomes zero, and hence the realized bankruptcy cost BC_τ is the difference between the default boundary K and the value of recovered risk-free bond $(1 - \bar{w})PV(D)$, where \bar{w} is the loss given default of all the debts. Before default happens, firm value is the sum of the debt value and

¹Please refer to [2] for the derivation of the first passage time distribution.

equity values, and the expected present value of bankruptcy cost. The expected present value of bankruptcy costs at time t is

$$BC_t = V_t^F - D_t - EQ_t. \quad (C.4)$$

BC_t defined in Equation (C.4) is a decreasing, convex function of the firm value V_t^F . It has the same properties as the bankruptcy cost defined in [4] and [5], i.e., BC_t satisfies the boundary conditions

$$\text{at } V_t^F = K, \quad BC_t = K - (1 - w)PV(D), \text{ and} \quad (C.5)$$

$$\text{as } V_t^F \rightarrow \infty, \quad BC_t \rightarrow 0. \quad (C.6)$$

C.2 Joint Distribution

In this section, we derive the joint distribution used in this paper. Let m_t and M_t be the running minimum and maximum, respectively, of a (μ, σ) Brownian motion \tilde{W}_t , which starts at zero, up to time t , i.e.,

$$m_t = \inf\{\tilde{W}_s : 0 \leq s \leq t\} \text{ and } M_t = \sup\{\tilde{W}_s : 0 \leq s \leq t\}. \quad (C.7)$$

Using the reflection principle and change of measure, it can be shown that the joint distribution of \tilde{W}_t and M_t is given by

$$\Pr\{\tilde{W}_t \leq x, M_t \leq y\} = \Phi\left(\frac{x - \mu t}{\sigma \sqrt{t}}\right) - \exp\left(\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{x - 2y - \mu t}{\sigma \sqrt{t}}\right), \quad (C.8)$$

for $y \geq 0$ and $x \leq y$, for details of the proof please see [2]. Since the process \tilde{W}_t start at zero, M_t is greater than zero for sure.

Following a similar discussion, we can get the joint distribution of \tilde{W}_t and m_t , which is

$$G_t(x, y)dx \equiv \Pr\{\tilde{W}_t \geq x, m_t \geq y\} = \Phi\left(\frac{-x + \mu t}{\sigma \sqrt{t}}\right) - \exp\left(\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{-x + 2y + \mu t}{\sigma \sqrt{t}}\right), \quad (C.9)$$

for $y \leq 0$ and $x \geq y$. Taking differentiation w.r.t. x , we have

$$g_t(x, y)dx \equiv \Pr\{\tilde{W}_t \in dx, m_t \geq y\} = \frac{1}{\sigma \sqrt{t}}\left[\phi\left(\frac{-x + \mu t}{\sigma \sqrt{t}}\right) - \exp\left(\frac{2\mu y}{\sigma^2}\right)\phi\left(\frac{-x + 2y + \mu t}{\sigma \sqrt{t}}\right)\right], \quad (C.10)$$

where $\phi(\cdot)$ is the standard normal probability density function.

For $m \leq 0$, we have

$$\Pr\{m_t \geq m\} = \Pr\{\tilde{W}_t \geq m, m_t \geq m\} = \Phi\left(\frac{-m + \mu t}{\sigma \sqrt{t}}\right) - \exp\left(\frac{2\mu m}{\sigma^2}\right)\Phi\left(\frac{m + \mu t}{\sigma \sqrt{t}}\right). \quad (C.11)$$

We have the distribution of m_t and hence the distribution of τ_m ,

$$\begin{aligned} \Pr\{\tau_m < t\} &= \Pr\{m_t < m\} = 1 - \Pr\{m_t \geq m\} \\ &= \Phi\left(\frac{m - \mu t}{\sigma \sqrt{t}}\right) + \exp\left(\frac{2\mu m}{\sigma^2}\right)\Phi\left(\frac{m + \mu t}{\sigma \sqrt{t}}\right). \end{aligned} \quad (C.12)$$

C.3 Calibration under Merton Framework

Under Merton framework, equity value is a European call option on firm value. Let F be the face value of firm's liability (zero coupon debt with maturity \bar{T}). Define leverage L by $L = F/V_t^F$. The time- t equity value is

$$EQ_t = V_t^F (\Phi(d_1^M) - e^{-r(\bar{T}-t)} L \Phi(d_2^M)), \quad (C.13)$$

where

$$d_1^M = (-\log(L) + (r + \sigma^F/2)(\bar{T} - t)) / (\sigma^F \sqrt{\bar{T} - t}), \quad (C.14)$$

$$d_2^M = d_1^M - \sigma^F \sqrt{\bar{T} - t}. \quad (C.15)$$

A European call option price under the Merton's framework is a compound option on firm value. The value of a call option with strike K^O and maturity T_O is

$$C_t = \frac{V_t^F}{N} (\Phi_2(a_1, d_1^M; \rho_M) - e^{-r(\bar{T}-t)} L \Phi_2(a_2, d_2^M; \rho_M)) - e^{-r(T_O-t)} K^O \Phi(a_2), \quad (C.16)$$

where

$$\rho_M = \sqrt{(T_O - t) / (\bar{T} - t)}, \quad (C.17)$$

$$a_1 = (\log(V_t^F / V_O^M) + (r + \sigma^F/2)(T_O - t)) / (\sigma^F \sqrt{T_O - t}), \quad (C.18)$$

$$a_2 = a_1 - \sigma^F \sqrt{T_O - t}. \quad (C.19)$$

V_O^M is the firm value threshold above which the call option will be exercised. Let $V_O^M = \alpha^M V_t^F$ and $K^O = \kappa^M EQ_t / N$.

Following the same derivation as in Section 4.3, we get the relationship between κ^M and α^M , given by

$$\kappa^M = \frac{\alpha^M \Phi(d_{1,T_O}^M) - e^{-r(\bar{T}-T_O)} L \Phi(d_{2,T_O}^M)}{\Phi(d_1^M) - e^{-r(\bar{T}-t)} L \Phi(d_2^M)}, \quad (C.20)$$

where

$$d_{1,T_O}^M = (-\log(L) + \log(\alpha^M) + (r + \sigma^F/2)(\bar{T} - T_O)) / (\sigma^F \sqrt{\bar{T} - T_O}),$$

$$d_{2,T_O}^M = d_{1,T_O}^M - \sigma^F \sqrt{\bar{T} - T_O}.$$

The equation used in calculating the relation between BS-implied volatility v and moneyness κ is

$$\begin{aligned} & \Phi_2(a_1, d_1^M; \rho_M) - e^{-r(\bar{T}-t)} L \Phi_2(a_2, d_2^M; \rho_M) - e^{-r(T_O-t)} \kappa^M (\Phi(d_1^M) - e^{-r(\bar{T}-t)} L \Phi(d_2^M)) \Phi(a_2) \\ &= (\Phi(d_1^M) - e^{-r(\bar{T}-t)} L \Phi(d_2^M)) (\Phi(l_1) - \kappa^M e^{-r(T_O-t)} \Phi(l_2)), \end{aligned} \quad (C.21)$$

where l_1 and l_2 are defined in equations (4.17) and (4.18).

As in Hull, Nelken and White[3], calibration can be done using put options, the results would be exactly the same as using call options due to call-put parity. The value of a put option with strike K^O and maturity T_O is

$$P_t = \frac{V_t^F}{N} (e^{-r(\bar{T}-t)} L\Phi_2(-a_2, d_2^M; -\rho_M) - \Phi_2(-a_1, d_1^M; \rho_M)) + e^{-r(T_O-t)} K^O \Phi(-a_2). \quad (C.22)$$

The equation used for calibration is

$$\begin{aligned} & e^{-r(\bar{T}-t)} L\Phi_2(-a_2, d_2^M; -\rho_M) - \Phi_2(-a_1, d_1^M; -\rho_M) + e^{-r(T_O-t)} \kappa^M (\Phi(d_1^M) - e^{-r(\bar{T}-t)} L\Phi(d_2^M)) \Phi(-a_2) \\ & = (\Phi(d_1^M) - e^{-r(\bar{T}-t)} L\Phi(d_2^M)) (\kappa^M e^{-r(T_O-t)} \Phi(-l_2) - \Phi(-l_1)). \end{aligned} \quad (C.23)$$

C.3.1 Extended Merton's Framework with Warrants in Capital Structure

In this section, we briefly discuss the Crouhy and Galai Model [1] and the valuation formula of warrants and equity under this framework. Following the discussion in Section 4.5.2, we present the calibration method under this extended Merton's framework. In the following discussion, we use the same notations as in Section 4.4 and follow the same exercise strategy described in Section 4.4, i.e., the warrants holders only exercise their warrants if firm value is larger than \bar{V}^F at time T_W .

Under the extended Merton framework, the warrant price value is given by the following equations

$$\begin{aligned} X_t^M &= e^{-r(T_W-t)} \int_{\log \alpha^W}^{\infty} \frac{V_t^{F*}}{N+M} ((e^x + \beta^W L) \Phi(d_{1,T_W}^M) - L e^{-r(\bar{T}-T_W)} \Phi(d_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \\ &\quad - e^{-r(T_W-t)} \frac{\beta^W L}{M} V_t^{F*} \Phi(-\log \alpha^W; -\mu_{T_W-t}, \sigma_{T_W-t}), \end{aligned} \quad (C.24)$$

where $\phi(x; \mu, \sigma)$ is the probability density function of normal distributed random variable with mean μ and standard deviation σ , $\Phi(x; \mu, \sigma)$ is the probability distribution function of normal distributed random variable with mean μ and standard deviation σ , $\mu_{T_W-t} = (r - 0.5\sigma^F)(T_W - t)$, $\sigma_{T_W-t} = \sigma^F \sqrt{T_W - t}$, and

$$d_{1,T_W}^M = (-\log(L) + \log(\beta^W L + e^x) + (r + \sigma^F/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}) \quad (C.25)$$

$$d_{2,T_W}^M = d_{1,T_W}^M - \sigma^F \sqrt{\bar{T} - T_W} \quad (C.26)$$

Common share value S_t is

$$\begin{aligned} S_t^M &= e^{-r(T_W-t)} \left(\int_{\log \alpha^W}^{+\infty} \frac{V_t^{F*}}{N+M} ((e^x + \beta^W L) \Phi(d_{1,T_W}^M) - L e^{-r(\bar{T}-T_W)} \Phi(d_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \right. \\ &\quad \left. + \int_{-\infty}^{\log \alpha^W} \frac{V_t^{F*}}{N} (e^x \Phi(\tilde{d}_{1,T_W}^M) - L e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \right), \end{aligned} \quad (C.27)$$

where

$$\tilde{d}_{1,T_W}^M = (-\log(L) + x + (r + \sigma^F/2)(\bar{T} - T_W))/(\sigma^F \sqrt{\bar{T} - T_W}) \quad (C.28)$$

$$\tilde{d}_{2,T_W}^M = \tilde{d}_{1,T_W}^M - \sigma^F \sqrt{\bar{T} - T_W}. \quad (C.29)$$

By following the exercise strategy described in Section 4.4, i.e., the warrants holders only exercise their warrants if firm value is larger than \bar{V}^F . Since $S_{T_W}^M(\bar{V}^F)$ is equal to K^W , the relationship between α^W and β^W is given by equation below

$$p((\alpha^W + \beta^W L)\Phi(d_{1,T_W}^{M,\alpha^W}) - Le^{-r(\bar{T}-T_W)}\Phi(d_{2,T_W}^{M,\alpha^W})) = \beta^W L \quad (C.30)$$

where

$$d_{1,T_W}^{M,\alpha^W} = (-\log(L) + \log(\alpha^W + \beta^W L) + (r + \sigma^F/2)(\bar{T} - T_W))/(\sigma^F \sqrt{\bar{T} - T_W}),$$

$$d_{2,T_W}^{M,\alpha^W} = d_{1,T_W}^{M,\alpha^W} - \sigma^F \sqrt{\bar{T} - T_W}.$$

Given all the parameters, the relationship between moneyness κ^W and firm value moneyness α^W can be found by using the definition of κ^W , i.e.,

$$\begin{aligned} \beta^W L = & \kappa^W (e^{-r(T_W-t)} (\int_{\log \alpha^W}^{+\infty} p((e^x + \beta^W L)\Phi(d_{1,T_W}^M) - Le^{-r(\bar{T}-T_W)}\Phi(d_{2,T_W}^M))\phi(x; \mu_{T_W-t}, \sigma_{T_W-t})dx \\ & + \int_{-\infty}^{\log \alpha^W} \frac{p}{1-p} (e^x \Phi(\tilde{d}_{1,T_W}^M) - Le^{-r(\bar{T}-T_W)}\Phi(\tilde{d}_{2,T_W}^M))\phi(x; \mu_{T_W-t}, \sigma_{T_W-t})dx)). \end{aligned} \quad (C.31)$$

For a given κ^W , we can solve for β^W and α^W . The Black-Scholes implied volatility v is connected to model parameters through the equation below.

$$\begin{aligned} & \left(p \int_{\log \alpha^W}^{+\infty} ((e^x + \beta^W L)\Phi(d_{1,T_W}^M) - Le^{-r(\bar{T}-T_W)}\Phi(d_{2,T_W}^M))\phi(x; \mu_{T_W-t}, \sigma_{T_W-t})dx \right. \\ & \quad \left. - \beta^W L F(-\log \alpha^W; -\mu_{T_W-t}, \sigma_{T_W-t}) \right) \div \left(p \int_{\log \alpha^W}^{+\infty} ((e^x + \beta^W L)\Phi(d_{1,T_W}^M) \right. \\ & \quad \left. - Le^{-r(\bar{T}-T_W)}\Phi(d_{2,T_W}^M))\phi(x; \mu_{T_W-t}, \sigma_{T_W-t})dx \right. \\ & \quad \left. + \frac{p}{1-p} \int_{-\infty}^{\log \alpha^W} (e^x \Phi(\tilde{d}_{1,T_W}^M) - Le^{-r(\bar{T}-T_W)}\Phi(\tilde{d}_{2,T_W}^M))\phi(x; \mu_{T_W-t}, \sigma_{T_W-t})dx \right) \\ & = \Phi(l_1^W) - e^{-r(T_W-t)} \kappa^W \Phi(l_2^W), \end{aligned} \quad (C.32)$$

where l_1^W and l_2^W are given by equations (4.48) and (4.49), respectively.

C.3.2 Option Price of Firm with Capital Structure including Warrants (Merton)

Following Section 4.5.3, we discuss the option price of firm with debt, equity and warrants in its capital structure. The discussion is under the extended Merton framework discussed in

Section C.3.1. We follow the notations in Section 4.5.3 and assume $T_O \leq T_W \leq \bar{T}$. Option holders only exercise the call option if firm value is larger than \bar{V}_O^F at T_O . Call option value is

$$\begin{aligned}
C_t &= e^{-r(T_O-t)} E_Q \left[(S^M(V_{T_O}^{F*}, \sigma^F) - K^O) I_{\{V_{T_O}^F \geq \bar{V}_O^F\}} \right] \\
&= e^{-r(T_O-t)} \int_{\log \alpha}^{+\infty} e^{-r(T_W-T_O)} V_t^{F*} e^y \left(\int_{\log \alpha^W-y}^{\infty} \frac{1}{N+M} ((e^x + \beta^W L e^{-y}) \Phi(d_{1,T_O}^M) \right. \\
&\quad - L e^{-y} e^{-r(\bar{T}-T_W)} \Phi(d_{2,T_O}^M)) \phi(x; \mu_{T_W-T_O}, \sigma_{T_W-T_O}) dx + \int_{-\infty}^{\log \alpha^W-y} \frac{1}{N} (e^x \Phi(\tilde{d}_{1,T_O}^M) \\
&\quad - L e^{-y} e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_{2,T_O}^M)) \phi(x; \mu_{T_W-T_O}, \sigma_{T_W-T_O}) dx \Big) \phi(y; \mu_{T_O-t}, \sigma_{T_O-t}) dy \\
&\quad - e^{-r(T_O-t)} \kappa S_t^M \Phi(-\log \alpha; -\mu_{T_O-t}, \sigma_{T_O-t}), \tag{C.33}
\end{aligned}$$

where

$$\begin{aligned}
d_{1,T_O}^M &= (-\log(L) + y + \log(\beta^W L e^{-y} + e^x) + (r + \sigma^{F^2}/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}) \\
d_{2,T_O}^M &= d_{1,T_O}^M - \sigma^F \sqrt{\bar{T} - T_W} \\
\tilde{d}_{1,T_O}^M &= (-\log(L) + y + x + (r + \sigma^{F^2}/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}) \\
\tilde{d}_{2,T_O}^M &= \tilde{d}_{1,T_O}^M - \sigma^F \sqrt{\bar{T} - T_W},
\end{aligned}$$

and S_t^M is given by equation (C.27).

The relationship between κ and α is given by $S_{T_O}^M(\bar{V}_O^F) = K^O = \kappa S_t^M(V^{F*})$. After simplification, it is

$$\begin{aligned}
\kappa &= e^{r(T_O-t)} \alpha \left(p \int_{\log \alpha^W - \log \alpha}^{+\infty} ((e^x + \beta^W \frac{L}{\alpha}) \Phi(\tilde{d}_{1,T_O}^M) - \frac{L}{\alpha} e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_{2,T_O}^M)) \phi(x; \mu_{T_W-T_O}, \sigma_{T_W-T_O}) dx \right. \\
&\quad + \frac{p}{1-p} \int_{-\infty}^{\log \alpha^W - \log \alpha} (e^x \Phi(\hat{d}_{1,T_O}^M) - \frac{L}{\alpha} e^{-r(\bar{T}-T_W)} \Phi(\hat{d}_{2,T_O}^M)) \phi(x; \mu_{T_W-T_O}, \sigma_{T_W-T_O}) dx \Big) \\
&\quad \div \left(p \int_{\log \alpha^W}^{+\infty} ((e^x + \beta^W L) \Phi(d_{1,T_W}^M) - L e^{-r(\bar{T}-T_W)} \Phi(d_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \right. \\
&\quad \left. + \frac{p}{1-p} \int_{-\infty}^{\log \alpha^W} (e^x \Phi(\tilde{d}_{1,T_W}^M) - L e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \right), \tag{C.34}
\end{aligned}$$

where

$$\begin{aligned}
\bar{d}_{1,T_O}^M &= (-\log(L) + \log(\alpha) + \log(\beta^W \frac{L}{\alpha} + e^x) + (r + \sigma^{F^2}/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}) \\
\bar{d}_{2,T_O}^M &= \bar{d}_{1,T_O}^M - \sigma^F \sqrt{\bar{T} - T_W} \\
\hat{d}_{1,T_O}^M &= (-\log(L) + \log(\alpha) + x + (r + \sigma^{F^2}/2)(\bar{T} - T_W)) / (\sigma^F \sqrt{\bar{T} - T_W}) \\
\hat{d}_{2,T_O}^M &= \hat{d}_{1,T_O}^M - \sigma^F \sqrt{\bar{T} - T_W}.
\end{aligned}$$

The Black-Scholes call option implied volatility ν is connected to model parameter inputs by the following equation

$$\begin{aligned}
& \int_{\log \alpha}^{+\infty} e^y \left(p \int_{\log \alpha^W - y}^{\infty} ((e^x + \beta^W L e^{-y}) \Phi(d_{1,T_0}^M) \right. \\
& - L e^{-y} e^{-r(\bar{T}-T_W)} \Phi(d_{2,T_0}^M)) \phi(x; \mu_{T_W-T_0}, \sigma_{T_W-T_0}) dx + \frac{p}{1-p} \int_{-\infty}^{\log \alpha^W - y} (e^x \Phi(\tilde{d}_{1,T_0}^M) \\
& - L e^{-y} e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_{2,T_0}^M)) \phi(x; \mu_{T_W-T_0}, \sigma_{T_W-T_0}) dx \Big) \phi(y; \mu_{T_0-t}, \sigma_{T_0-t}) dy \\
& - e^{-r(T_0-t)} \kappa \Phi(-\log \alpha; -\mu_{T_0-t}, \sigma_{T_0-t}) \left(p \left(\int_{\log \alpha^W}^{+\infty} ((e^x + \beta^W L) \Phi(d_{1,T_W}^M) \right. \right. \\
& - L e^{-r(\bar{T}-T_W)} \Phi(d_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \\
& + \frac{p}{1-p} \int_{-\infty}^{\log \alpha^W} (e^x \Phi(\tilde{d}_{1,T_W}^M) - L e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \Big) \\
& = (\Phi(l_1) - \kappa e^{-r(T_0-t)} \Phi(l_2)) \left(p \left(\int_{\log \alpha^W}^{+\infty} ((e^x + \beta^W L) \Phi(d_{1,T_W}^M) \right. \right. \\
& - L e^{-r(\bar{T}-T_W)} \Phi(d_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \\
& + \frac{p}{1-p} \int_{-\infty}^{\log \alpha^W} (e^x \Phi(\tilde{d}_{1,T_W}^M) - L e^{-r(\bar{T}-T_W)} \Phi(\tilde{d}_{2,T_W}^M)) \phi(x; \mu_{T_W-t}, \sigma_{T_W-t}) dx \Big). \tag{C.35}
\end{aligned}$$

Bibliography

- [1] Michel Crouhy and Dan Galai. The interaction between the financial and investment decisions of the firm: the case of issuing warrants in a levered firm. *Journal of Banking & Finance*, 18(5):861 – 880, 1994.
- [2] J Michael Harrison. *Brownian motion and stochastic flow systems*. Wiley New York, 1985.
- [3] John Hull, Izzy Nelken, and Alan White. Mertons model, credit risk, and volatility skews. *Journal of Credit Risk*, 1(1), 2004.
- [4] Hayne E. Leland. Corporate debt value, bond covenants, and optimal capital structure. *The Journal of Finance*, 49(4):1213–1252, 1994.
- [5] Hayne E. Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. *The Journal of Finance*, 51(3):987–1019, 1996.
- [6] Xinghua Zhou and R. Mark Reesor. Misrepresentation and capital structure: Quantifying the impact on corporate debt value. *Journal of Corporate Finance*, 34:293 – 310, 2015.

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